Simultaneous Search in Competitive Markets

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Abstract

This paper examines simultaneous search behavior of a firm for the highest selling price and the lowest wage rate. It begins by formulating simultaneous search problem of a competitive firm, and then characterizes the optimal search. The comparative static results and some welfare implications follow. In the end, for a possible extension, simultaneous search problem of a monopolist is considered.

Keywords: simultaneous search, competitive firm, fixed-sample-size search

1 Introduction

Through his seminal article in 1961, Stigler (1961) called attentions of the profession to studies on individual behavior of search. By now, the literature has become huge, diverting itself in several directions. Although it is potentially extendable to other contexts, most models on personal search have been originally proposed as either consumer search or job search. Few have focused on a firm's search. This paper studies a firm's search.

Consumer search and job search are the two most active areas of research on individual search. Consumer search is often seen as search for the lowest price while job search is seen as search for the highest price. So, these are in some sense two extreme cases. As discussed below, a firm's search contains both aspects. Furthermore, it turns out shortly that a firm's search behavior is different from a consumer's, even though both of them search for lower prices.

There has been several different search strategies that searchers may take. They include sequential strategy, fixed-sample-size (fss) strategy, and their hybrid, optimal search strategy, among others.[1] Every one of these has its own advantage. So, it is not possible to

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[1] For instance, see Rothschild (1974) and Weitzman (1971) for sequential search, Manning and Morgan (1982) and Morgan (1983) for fss search and Morgan and Manning (1985) and Chade and Smith (2006) for optimal search.

claim in general which dominates which search rules.[2] This paper restricts its attention to fss search in which a searcher determines how much to search before she starts searching. The advantage of this search rule includes that a searcher may collect information more quickly. Note, however, that our attention to this search strategy does not means that we insist that firms follow this search rule more than others. No doubt there are some cases in which another rule fits better.

The model of concern in this paper is as follows. Consider a firm that produces a single commodity y by using inputs $x \in \mathbb{R}^l$ where $\mathbb{R}^l$ is an l-dimensional real space with some integer $l \geq 1$. Unlike standard firm models, suppose that there are several consumer markets for commodity y and several factor markets for some input x_i. Moreover, suppose that each market is characterized by a parameter and that the firm knows the distributions of the parameter values but is ignorant of the exact location of each value. That is, it knows existing market types and the frequencies, but cannot tell which market is of which type. The firm, however, can "visit" any of those markets at some cost and see its complete characteristics. To be more specific, by canvassing a consumer market for commodity y (or a factor market for input x_i), he can learn the selling price (or the wage rate) in that market.

In such a setup, a firm now involves two problems to solve for his ultimate purpose of profit maximization: a conventional problem on production and an additional problem on search. Here, not only how many units of x and y to use and produce, but also where to buy x_i and where to sell y come into consideration.

The objective in this paper is to formally formulate a firm's problem of search in this setup and to examine an optimal search behavior.

The paper proceeds as follows. Section 2 formally constructs a firm's problem of simultaneous fss search in some simplified setting. In section 3, we characterize an optimal simultaneous search by solving the firm problem. Sections 4 and 5 consider comparative statics and production. In section 6, some welfare implications of the firm's search is discussed. To conclude, section 7 remarks some extensions and limitations of our model.

2 Firm's Simultaneous Search Problem

The model of concern in this paper is based on the following assumptions.

Assumption 1. A firm, whose technology is represented by a function $\zeta \colon \mathbb{R}^2 \to \mathbb{R}$, produces a commodity y by using $x = (x_1, x_2)$ as inputs.

[2] For more discussion on this matter, see for example Morgan and Manning (1985).

Assumption 2. There is a *set* of competitive consumer markets for commodity y, offering various selling prices p as a whole.

Assumption 3. There is a *set* of competitive factor markets for input x_1, offering various input prices w_1 as a whole.

Assumption 4. There is only one factor market for input x_2, which is competitive and offers an input price w_2.

Assumption 5. The distributions of selling prices for y and input prices for x_1 are known, but the exact location of any particular price is not *except those of the lowest selling price and the highest input price.*

Assumption 6. The firm learns the selling price in a consumer market at cost c_λ and the input price in a factor market for input x_1 at cost c_θ.

Assumptions 1–6 give a simple setup for our analysis on a firm's simultaneous search behavior. Of course, we can obtain more general setups under weaker assumptions. For example, search can involve more than three prices; neither consumer markets nor factor markets are necessarily competitive; or price locations can be completely uncertain at the outset. Some of those cases will be discussed later.

Before proceeding, it is better for concreteness to propose a possible interpretation of this economy. The consumer markets consist of one "domestic" and many other "foreign" markets. The firm under consideration is an incumbent of this domestic market. The selling price he undergoes in this domestic market is very low, so he wants to "export" his product to a more profitable market. He knows the ranges of selling prices offered in foreign markets, but cannot tell which price belongs to which market. Fortunately, such information is obtainable through "marketing research" whose cost is c_λ for each market.

While the firm exports his product to a foreign market, transportation cost becomes his concern. If x_1 is an input for physical distribution of commodity y, then the input price for x_1 is the "distribution cost." Suppose that several companies offer this distribution service x_1 at different prices. The firm under consideration knows one of the companies. Unfortunately, this company is inferior in exporting y and charges the highest price among all distribution firms, since it is specialized to domestic transportation. Once again, he knows the distribution of asking prices for this service, but is ignorant of the exact locations except one. In exchange of c_θ per company, he can learn the asking prices of other companies. Since it improves its physical distribution system, the firm's search for the lowest input price may be considered as "cost-reducing process R&D".

The following notations are used throughout the paper. The lowest selling price for commodity y is written by $\underline{\lambda}p_0$. The highest factor price for input x_1 is written by $\overline{\theta}w_0$. With these, the associated price dispersions for commodity y and input x_1 are denoted by $[\underline{\lambda}p_0, \overline{\lambda}p_0]$ and $[\underline{\theta}w_0, \overline{\theta}w_0]$. By construction, we let $\lambda \geq 1$ with $\underline{\lambda} = 1$ and let $\theta \leq 1$ with $\overline{\theta} = 1$ where $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ and $\theta \in [\underline{\theta}, \overline{\theta}]$. We call p_0 and w_0 as "reference prices" while λ and θ as a "premium" or a "discount factor", respectively.

Furthermore, the underlying cumulative distribution functions (cdf) for premium λ and discount factor θ are written as F and G, respectively. Together with p_0 and w_0, they also provide the price distributions for commodity y and input x_1. The probability density functions (pdf) of F and G are given in the associated lower-case letters, f and g.

Now we start modeling a firm's simultaneous search problem in this setting. Here, the firm's problem consists of two components: production and search. That is, he decides not only how many units of x and y to use and to produce, but also from which market to buy x_1 and to which market to sell y in order to maximize his expected profit. We consider a two-stage setup, where he conducts searches in stage 1 and begins production in stage 2. From the firm's viewpoint, λ and θ are random variables with known probabilities. So, the search problem simplifies into a problem of choosing the numbers of observations n_λ and n_θ, which we refer to as intensities of search.

Begin by stage 2. The firm's problem in stage 2 is a conventional production problem. That is, given a selling price λp_0 and input prices θw_0 and w_2, it solves

$$\begin{aligned} \max_{y, x_1, x_2} \quad & \lambda p_0 y - \theta w_0 x_1 - w_2 x_2 \\ \text{subject to} \quad & y = \zeta(x_1, x_2). \end{aligned}$$

The objective function $\lambda p_0 y - \theta w_0 x_1 - w_2 x_2$ is the direct profit.[3] The solution to this second-stage problem is a set of supply and factor demand functions:

$$\begin{aligned} & y^*(\lambda p_0, \theta w_0, w_2), \\ & x_i^*(\lambda p_0, \theta w_0, w_2) \quad i = 1, 2. \end{aligned}$$

Substituting these into the direct profit gives the indirect profit function

$$\pi(\lambda p_0, \theta w_0, w_2) = \lambda p_0 y^*(\lambda p_0, \theta w_0, w_2) - \theta w_0 x_1^*(\lambda p_0, \theta w_0, w_2) - w_2 x_2^*(\lambda p_0, \theta w_0, w_2).$$

Note the basic properties of the indirect profit function that π is nondecreasing in selling price λp_0 and nonincreasing in input prices θw_0 and w_2.

[3]Please note that the profit does not include either $n_\lambda c_\lambda$ or $n_\theta c_\theta$. It is because those costs are already sunk.

Go to stage 1. First, consider a situation after search. By conducting search of intensity $n = (n_\lambda, n_\theta)$, the firm observes n_λ premiums, $\lambda_1, \cdots, \lambda_{n_\lambda}$, and n_θ discount factors, $\theta_1, \cdots, \theta_{n_\theta}$. For each pair of (λ_i, θ_j), he computes the direct profit and then solves the maximization problem to obtain the indirect profit function $\pi(\lambda_i p_0, \theta_j w_0, w_2)$. Here, i and j are integers with $1 \le i \le n_\lambda$ and $1 \le j \le n_\theta$. In sum, he has $n_\lambda n_\theta$ profit functions to compare. For his purpose of profit maximization, it is clear that he prefers λ_m and θ_m defined by

$$\lambda_m = \max\{\lambda_1, \ldots, \lambda_{n_\lambda}\} \tag{1}$$

$$\theta_m = \min\{\theta_1, \ldots, \theta_{n_\theta}\}. \tag{2}$$

since $\pi(\lambda_m p_0, \theta_m w_0, w_2) \ge \pi(\lambda_i p_0, \theta_j w_0, w_2)$ for any i and j.

Before going to search, he can compute this π for each pair of $(\lambda_m, \theta_m) \in [\underline{\lambda}, \overline{\lambda}] \times [\underline{\theta}, \overline{\theta}]$. The pdf of λ_m, which is the maximum of a sample of n_λ independent observations from an identical population with the pdf f, is

$$f^*(\lambda_m | n_\lambda) = n_\lambda [F(\lambda_m)]^{n_\lambda - 1} f(\lambda_m), \quad n_\lambda \ge 1. \tag{3}$$

The pdf of θ_m, which is the minimum of a sample of n_θ independent observations from an identical population with the pdf g, is

$$g^*(\theta_m | n_\theta) = n_\theta [1 - G(\theta_m)]^{n_\theta - 1} g(\theta_m), \quad n_\theta \ge 1. \tag{4}$$

Let F^* and G^* be the corresponding cdf's. Let $n = (n_\lambda, n_\theta)$ and $\alpha = (c_\lambda, c_\theta, p_0, w_0, w_2)$. Then the firm's expected direct search profit net of search costs equals

$$H(n; \alpha) = E[\pi(\lambda_m p_0, \theta_m w_0, w_2) | n] - n_\lambda c_\lambda - n_\theta c_\theta \tag{5}$$

where E stands for the expectation taken with respect to distributions F^* and G^*:

$$E[\pi(\lambda_m p_0, \theta_m w_0, w_2) | n] = \int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m | n_\lambda) g^*(\theta_m | n_\theta) \, d\lambda_m \, d\theta_m. \tag{6}$$

$E[\pi(\cdot)|n]$ is termed as the expected indirect profit. The firm's search problem is to select intensity n so at to maximize the expected direct search profit net of search cost $H(n; \alpha)$ in (5).[4]

[4]Note that $H(n)$ in (5) is well-defined under the stated assumptions. In particular, by Assumptions 5 the firm knows the exact locations of $\underline{\lambda}$ and $\overline{\theta}$ at the outset. So, even if n_λ=0 and/or $n_\theta = 0$, he has $y^*(\underline{\lambda} p_0, \overline{\theta} w_0, w_2)$ and $x_i^*(\underline{\lambda} p_0, \overline{\theta} w_0, w_2)$ and obtains $\pi(\underline{\lambda} p_0, \overline{\theta} w_0, w_2)$. It is obvious for other nonnegative integers. Hence $H(n)$ is well-defined. One may want to weaken this assumption by imposing complete ignorance of price locations. In that case, some adjustments in defining f^* and g^* are necessary for a well-defined $H(n)$. For necessary adjustments that results in a well-defined $H(n)$ (as a Lebesgue-Stieltjes integral), see Manning and Morgan (1982), pp.205-6.

3 The Solution of Firm's Problem

This section characterizes an optimal simultaneous search of this firm, by solving the problem set above. Before doing so, however, let us introduce some additional assumptions on technical grounds.

The first one is about intensity n. The firm's search problem is to select n so as to attain a maximized $H(n)$ in (5). Precisely speaking, this is an integer-valued problem. An optimal n, written as n^*, belongs to $\mathbb{Z}_+^2$, where $\mathbb{Z}_+$ is a set of nonnegative integers. However, it becomes analytically more convenient if n can be treated as continuous variables. For this reason, we set the following assumption.

Assumption 7. n is a nonnegative two-dimensional real.

Under this additional assumption, the firm's simultaneous search problem now becomes to choose $n \in \mathbb{R}_+^2$ that maximizes (5).[5]

The interior solutions may be of particular interest. To this end, let production function ζ, indirect profit function π and expected direct search profit net of search cost H satisfy the following assumptions.

Assumption 8. Let $\underline{x_i} = x_i^*(\underline{\lambda} p_0, \overline{\theta} w_0, w_2)$ for $i = 1, 2$. Then, ζ is differentiable in x and satisfies

$$\lim_{x_i \to \underline{x_i}} \frac{\partial \zeta(x)}{\partial x_i} = +\infty, \quad i = 1, 2.$$

Assumption 9. $\pi(\lambda_m p_0, \theta_m w_0, w_2)$ is differentiable in $\lambda_m p_0$, $\theta_m w_0$ and w_2.

Assumption 10. $H(n; \alpha)$ is twice differentiable in n and α.

Assumption 8 makes productions at $x \in (\underline{x_1}, \infty) \times (\underline{x_2}, \infty)$ essential and ensures positive search intensities. Assumptions 8 and 9 together make calculus approach relevant to this problem.

The following two propositions characterize the profit-maximizing search intensity of the firm under the stated assumptions.

Proposition 1. *An optimal search intensity $n^* = (n_\lambda^*, n_\theta^*)$ satisfies*

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) \frac{\partial f^*(\lambda_m | n_\lambda)}{\partial n_\lambda} g^*(\theta_m | n_\theta)\, d\lambda_m\, d\theta_m = c_\lambda \tag{7}$$

[5]This is convention in the profession on this subject. One justification for it is that errors resulting from this assumption are not great in the sense that two n^*'s, one from real-valued problem and one from original, differ by less than 1 if H is strictly concave. See Manning and Morgan (1982), p.206.

and

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m|n_\lambda) \frac{\partial g^*(\theta_m|n_\theta)}{\partial n_\theta} \, d\lambda_m \, d\theta_m = c_\theta. \tag{8}$$

Proof. For the interior solution, necessarily $DH = 0$ holds. (7) and (8) are just this rearranged. □

Proposition 1 states that the optimal intensity of search equates the marginal benefit of search with its marginal cost. The LHS of (7) is $\partial E[\pi|n_\lambda, n_\theta]/\partial n_\lambda$, meaning the marginal expected profit resulting from additional increment of search for λ. The RHS is its corresponding cost. Similarly, the LHS of (8), $\partial E[\pi|n_\lambda, n_\theta]/\partial n_\theta$ gives the marginal expected after-search profit arising from additional search for θ. The RHS is its cost.

Proposition 2. *(7) and (8) represent a local optimal search intensity if, for $n = n^*$,*

$$\left[\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) \frac{\partial^2 f^*(\lambda_m|n_\lambda)}{\partial {n_\lambda}^2} g^*(\theta_m|n_\theta) \, d\lambda_m \, d\theta_m\right]$$
$$\cdot \left[\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m|n_\lambda) \frac{\partial^2 g^*(\theta_m|n_\theta)}{\partial {n_\theta}^2} \, d\lambda_m \, d\theta_m\right]$$
$$> \left[\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) \frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda} \frac{\partial g^*(\theta_m|n_\theta)}{\partial n_\theta} \, d\lambda_m \, d\theta_m\right]^2. \tag{9}$$

Further, if (9) holds for all $n \in \mathbb{R}^2_+$, then (7) and (8) represent global optimum.

Proof. In Appendix 1. □

The condition that (9) imposes on the optimal search is that, in some neighborhood of n^*, "own effects" on marginal benefits of search must outweigh their "cross effects".

We cannot determine whether the inequality holds in general. From Corollaries 5 and 6 in Appendix 2, we know that the LHS of (9) is positive and the RHS is nonnegative under Assumption 8. Beyond that, more specific formula for f and g are necessary.

4 Comparative Statics

The firm's problem in our setting involves two stages: stage of search and stage of production. Of particular interest is how a firm's search behavior changes if the environment of search or the environment of second-stage production changes. These are problems concerning comparative statistics.

Suppose that in some neighborhood of n^*, (9) holds. Then, by Proposition 2, a (locally) optimal search intensity exists. Moreover, since this implies that D^2H (the Hessian of H)

is nonzero, (in principle) we can locally solve implicit relations (7) and (8) for the optimal n as functions of parameters. That is, with $\alpha = (c_\lambda, c_\theta, p_0, w_0, w_2)$

$$n_\lambda^* = n_\lambda^*(\alpha) \tag{10}$$

$$n_\theta^* = n_\theta^*(\alpha). \tag{11}$$

Substituting these into (5) provides the firm's expected indirect search profit net of search costs

$$\phi(\alpha) = H(n^*(\alpha);\alpha) = E[\pi(\lambda_m p_0, \theta_m w_0, w_2)|n^*(\alpha)] - n_\lambda^*(\alpha)c_\lambda - n_\theta^*(\alpha)c_\theta. \tag{12}$$

In what follows, we examine the effect of changes in parameter α on the optimal search intensity n^*. The meaning of changes in c_λ, c_θ and w_2 is clear. An increase in p_0 means that the distribution of selling prices is scaled up by a constant proportion. Of course, this increases both of the mean and variance. So, it is a special kind of increasing risk. The implication of an increase in w_0 is similar.

Proposition 3. *Suppose that (9) holds for $n = n^*$. Then,*

(i) the optimal search intensity for the highest selling discount factor λ_m is nondecreasing with scale changes in the distribution of selling prices. That is,

$$\frac{\partial n_\lambda^*}{\partial p_0} \geq 0. \tag{13}$$

(ii) The optimal search intensity for the lowest discount factor θ_m of input x_1 is independent of scale changes in the distribution of selling prices p_0. That is,

$$\frac{\partial n_\theta^*}{\partial p_0} = 0. \tag{14}$$

(iii) The optimal search intensity for the highest selling discount factor λ_m is independent of scale changes in the distribution of discount factors of input x_1. That is,

$$\frac{\partial n_\lambda^*}{\partial w_0} = 0. \tag{15}$$

(iv) The optimal search intensity for the lowest discount factor θ_m of input x_1 is nondecreasing with scale changes in the distribution of discount factors of input x_1. That is,

$$\frac{\partial n_\theta^*}{\partial w_0} \geq 0. \tag{16}$$

(v) The optimal search intensity for the highest selling discount factor λ_m is independent of changes in prices of input x_2. That is,

$$\frac{\partial n_\lambda^*}{\partial w_2} = 0. \tag{17}$$

(vi) The optimal search intensity for the lowest discount factor θ_m of input x_1 is independent of changes in prices of input x_2. That is,

$$\frac{\partial n_\theta^*}{\partial w_2} = 0. \tag{18}$$

(vii) An increase in search cost for the highest selling discount factor decreases the optimal amount of search for the highest selling discount factor λ_m. That is,

$$\frac{\partial n_\lambda^*}{\partial c_\lambda} < 0. \tag{19}$$

(viii) An increase in search cost for the lowest discount factor of input x_1 decreases the optimal amount of search for the lowest discount factor θ_m of input x_1. That is,

$$\frac{\partial n_\theta^*}{\partial c_\theta} < 0. \tag{20}$$

(ix) The optimal search intensity for the highest selling discount factor λ_m is independent of changes in search costs for the lowest discount factor of input x_1. That is,

$$\frac{\partial n_\lambda^*}{\partial c_\theta} = 0. \tag{21}$$

(x) The optimal search intensity for the lowest discount factor θ_m of input x_1 is independent of changes in search costs for the highest selling discount factor. That is,

$$\frac{\partial n_\theta^*}{\partial c_\lambda} = 0. \tag{22}$$

Proof. In Appendix 1. □

Parts (vii) and (viii) of Proposition 3 are nothing but "the law of demand" for firm's search by viewing "price information" as one particular good. Parts (i) and (iv) say that increased risk in the distributions of selling prices and input prices raises the associated optimal search level. The rest of the proposition shows insensitivity of firm's search behaviors to the other parameters.

Note that in consumer search, an increase in the list price of a non-searched-for commodity will increase the optimal amount of search for a searched-for-commodity if the two commodities are substitutes (Manning and Morgan, 1982, p.210). So, the insensitivity of firm's search in part (vi) shows a difference in search behavior between sellers and buyers even though both of them seek lower prices.

5 Expected Production in Search Economy

In this section, we examine the benefit of search to a firm's expected production in this search economy.

Hotelling's lemma tells us how to obtain the supply function $y^*(\cdot)$ and the factor demand function $x_i^*(\cdot)$ from a profit function $\pi(\cdot)$, given a commodity price p and an input price w_i. Without search, it is

$$\begin{aligned} \frac{\partial \pi(p, w_i)}{\partial p} &= y^*(p, w_i), \\ -\frac{\partial \pi(p, w_i)}{\partial x_i} &= x_i^*(p, w_i). \end{aligned}$$

The counterpart of this derivative property in our setup is

$$\begin{aligned} \frac{\partial \phi(p_0, w_0, w_2)}{\partial p_0} &= \int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \lambda_m y^*(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m | n_\lambda) g^*(\theta_m | n_\theta)\, d\lambda_m\, d\theta_m, \\ -\frac{\partial \phi(p_0, w_0, w_2)}{\partial w_0} &= \int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \theta_m x_1^*(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m | n_\lambda) g^*(\theta_m | n_\theta)\, d\lambda_m\, d\theta_m, \\ -\frac{\partial \phi(p_0, w_0, w_2)}{\partial w_2} &= \int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} x_2^*(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m | n_\lambda) g^*(\theta_m | n_\theta)\, d\lambda_m\, d\theta_m. \end{aligned}$$

Given a reference price p_0, a reference input price w_0 and an input price w_2, the expressions in the RHS are interpreted as the expected supply and factor demand functions. It says that, for a non-searched-for commodity such as input x_2, taking expectation of the factor demand function $x_2^*(\cdot)$ delivers the expected factor demand function, but this does not apply for searched-for commodities. To obtain the expected supply function, the supply function $y^*(\cdot)$ must be weighted by its searched-for premium λ_m while its expectation being taken. Similarly, to represent the expected factor demand function, the factor demand function $x_1^*(\cdot)$ has to be weighted by its searched-for discount factor θ_m while its expectation being taken.

The next proposition describes a relationship between optimal search and optimal expected production.

Proposition 4. *Suppose that (9) holds. Then, search brings positive marginal benefit to expected production. That is, the expected marginal production benefit of search is positive.*

Proof. For the proof, it suffices to show

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \lambda_m y^*(\lambda_m p_0, \theta_m w_0, w_2) \frac{\partial f^*(\lambda_m | n_\lambda^*)}{\partial n_\lambda} g^*(\theta_m | n_\theta^*)\, d\lambda_m\, d\theta_m > 0, \tag{23}$$

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \theta_m x_1^*(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m | n_\lambda^*) \frac{\partial g^*(\theta_m | n_\theta^*)}{\partial n_\theta} \, d\lambda_m \, d\theta_m > 0.$$

These follow instantly from the Envelope Theorem, and Corollaries 3 and 4. For example, see (23). By the optimality of the supply function $y^*(\cdot)$, the Envelope Theorem gives $\frac{\partial(\lambda_m p_0 y^*)}{\partial \lambda_m} = p_0 y^*$ that is positive under Assumption 8. Corollary 3 in Appendix 2 then implies (23). The other claim can be established in a similar manner. □

Production does not begin until search is complete. Furthermore, production utilizes all the findings obtained through search. Proposition 4 reflects this sequence of the firm's move. Note

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} y^*(\lambda_m p_0, \theta_m w_0, w_2) \frac{\partial f^*(\lambda_m | n_\lambda^*)}{\partial n_\lambda} g^*(\theta_m | n_\theta^*) \, d\lambda_m \, d\theta_m = 0, \tag{24}$$

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} x_1^*(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m | n_\lambda^*) \frac{\partial g^*(\theta_m | n_\theta^*)}{\partial n_\theta} \, d\lambda_m \, d\theta_m = 0.$$

These contrasts suggest why simply taking the expectation of $y^*(\cdot)$ and $x_1^*(\cdot)$ is not enough to represent the expected supply and factor demand functions.

6 Some Welfare Implications

This section considers some welfare implications of simultaneous search. Specifically, we characterize welfare-maximizing simultaneous search and then figure out the following issues. (1) Does profit-maximizing search maximize social welfare? (2) If not, how does it different from welfare-maximizing search? (3) At least, does it make society better-off?

To address these, we need clarify what we mean by "society". In what follows, we interpret both of inputs x_1 and x_2 as labor.

Assumption 11. Society is composed of a firm under consideration, its consumers and its employees (the laborers of x_1 and x_2).

Assumption 12. Each commodity market has an identical representative consumer and each labor market of x_i has an identical representative laborer.

A representative consumer's indirect utility and that of a representative laborer of x_i are denoted, respectively, by

$$v_c(p) = v_c(p, I_c) = \max_{y,z} \; u(y, z) \quad \text{subject to} \;\; py + z = I_c,$$

$$v_i(w_i) = v_i(w_i, I_i, L) = \max_{z,x_i} \; u(z, L - x_i) \quad \text{subject to} \;\; z = w_i x_i + I_i, \tag{25}$$

where I_c and I_i are exogenous initial endowments of a consumer and a laborer of x_i respectively, z, a numéraire and L, total time available to each laborer.[6]

Assumption 13. Social surplus is a sum of firm's profit and the indirect utility of its consumers and employees.

Under these additional assumptions, social surplus is simply written as

$$\pi(\lambda_m p_0, \theta_m w_0, w_2) + v_c(\lambda_m p_0) + v_1(\theta_m w_0) + v_2(w_2).$$

In this simplified setting, we discuss the welfare implications of simultaneous search.

6.1 The Welfare-Maximizing Simultaneous Search Problem

First, we formulate the problem of welfare-maximizing simultaneous search. The expected social welfare net of search cost is written as

$$W(n;\alpha) \;=\; H(n;\alpha) + \int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0) f^*(\lambda_m|n_\lambda)\, d\lambda_m + \int_{\underline{\theta}}^{\overline{\theta}} v_1(\theta_m w_0) g^*(\theta_m|n_\theta)\, d\theta_m. \quad (26)$$

The problem of welfare-maximizing simultaneous search is to maximize (26) with respect to n.

For technical reasons, we add two conventional assumptions on functions W, v_c and v_i.

Assumption 14. $W(n,\alpha)$ is twice differentiable in n and α.

Assumption 15. $v_c(\cdot)$ and $v_i(\cdot)$ are both differentiable in their arguments.

The next two propositions parallel Propositions 1 and 2. They characterize a socially optimal intensity of simultaneous search.

Proposition 5. *Suppose (9) holds. The welfare-maximizing search intensity* $n^{**} = (n_\lambda^{**}, n_\theta^{**})$ *satisfies*

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) \frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda} g^*(\theta_m|n_\theta)\, d\lambda_m\, d\theta_m = c_\lambda - \int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0) \frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda}\, d\lambda_m, \quad (27)$$

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m|n_\lambda) \frac{\partial g^*(\theta_m|n_\theta)}{\partial n_\theta}\, d\lambda_m\, d\theta_m = c_\theta - \int_{\underline{\theta}}^{\overline{\theta}} v_1(\theta_m w_0) \frac{\partial g^*(\theta_m|n_\theta)}{\partial n_\theta}\, d\theta_m. \quad (28)$$

[6] It is noted here for later reference that we are imposing Walras's law (or equality constraint) in each maximization problem. One implication of this is that marginal utility of income is positive.

Proof. For the interior solution, $DW = 0$ necessarily holds. These are just this rearranged. $\square$

Proposition 5 says that welfare-maximizing search equates marginal expected *social* benefit of search to its marginal expected *social* cost. The difference from Proposition 1 that characterizes profit-maximizing search is that marginal expected costs of search now include costs incurred by consumers and employees. It is noted that Corollaries 3 and 4 in Appendix 2 imply

$$\int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0) \frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda}\, d\lambda_m \leq 0,$$
$$\int_{\underline{\theta}}^{\overline{\theta}} v_1(\theta_m w_0) \frac{\partial g^*(\theta_m|n_\theta)}{\partial n_\theta}\, d\theta_m \leq 0$$

since $v_c(\cdot)$ is nonincreasing in $\lambda_m p_0$ while $v_1(\cdot)$ is nondecreasing in $\theta_m w_0$.

Proposition 6. *(27) and (28) represent (local) welfare-maximizing search intensity if, at* n^{**}*,*

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) \frac{\partial^2 f^*(\lambda_m|n_\lambda)}{\partial {n_\lambda}^2} g^*(\theta_m|n_\theta)\, d\lambda_m\, d\theta_m < - \int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0) \frac{\partial^2 f^*(\lambda_m|n_\lambda)}{\partial {n_\lambda}^2}\, d\lambda_m, \tag{29}$$

$$\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\lambda_m p_0, \theta_m w_0, w_2) f^*(\lambda_m|n_\lambda) \frac{\partial^2 g^*(\theta_m|n_\theta)}{\partial {n_\theta}^2}\, d\lambda_m\, d\theta_m < - \int_{\underline{\theta}}^{\overline{\theta}} v_1(\theta_m w_0) \frac{\partial^2 g^*(\theta_m|n_\theta)}{\partial {n_\theta}^2}\, d\theta_m, \tag{30}$$

and

$$\begin{aligned}
&\left[\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\cdot) \frac{\partial^2 f^*(\lambda_m|n_\lambda)}{\partial {n_\lambda}^2} g^*(\theta_m|n_\theta)\, d\lambda_m\, d\theta_m + \int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0) \frac{\partial^2 f^*(\lambda_m|n_\lambda)}{\partial {n_\lambda}^2}\, d\lambda_m\right] \\
&\quad \cdot \left[\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\cdot) f^*(\lambda_m|n_\lambda) \frac{\partial^2 g^*(\theta_m|n_\theta)}{\partial {n_\theta}^2}\, d\lambda_m\, d\theta_m + \int_{\underline{\theta}}^{\overline{\theta}} v_1(\theta_m w_0) \frac{\partial^2 g^*(\theta_m|n_\theta)}{\partial {n_\theta}^2}\, d\theta_m\right] \\
&\qquad > \left[\int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\cdot) \frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda} \frac{\partial g^*(\theta_m|n_\theta)}{\partial n_\theta}\, d\lambda_m\, d\theta_m\right]^2.
\end{aligned} \tag{31}$$

Further, if (29)-(31) are true for all $n \in \mathbb{R}_+^2$*, (27) and (28) give global maximum.*

Proof. In Appendix 1. $\square$

Notice first that none of these holds in general. For both of (29) and (30), Theorem 2 and Corollaries 2, 5 and 6 in Appendix 2 imply that the both sides of the inequalities may be negative. For (31), the both sides may be positive.

The economic meaning of these conditions is given as follows. (29) says that, in expectation, firm's marginal profit must be more sensitive to search than consumer's marginal utility. Similarly, (30) states that, in expectation, firm's marginal profit must be more sensitive to search than employee's marginal utility. One possible interpretation for (31) is that "own effects" on marginal benefits of search in society must outweigh their "cross effects" in society.

6.2 Social Sub-optimality of Firm's Search

Suppose that (29)–(31) are all satisfied. Then, the next proposition answers our first question.

Proposition 7. *Suppose that (9) and (29)–(31) hold. Then, profit-maximizing search does not maximize social surplus.*

Proof. In Appendix 1. □

The next proposition exhibits how profit-maximizing search differs from welfare-maximizing search.

Proposition 8. *Suppose that (9) and (29)-(31) hold. Let n^* and n^{**} be intensities of profit-maximizing search and welfare-maximizing search, respectively. Then, $n^* > n^{**}$ holds.*

Proof. In Appendix 1. □

Proposition 8 states that whether it is for the lowest prices or the highest prices, a profit-maximizing firm searches too much relative to social optimum.

6.3 Does Search Make Society Better-Off?

Search changes a resulting economy. Proposition 7 shows that firm's profit-maximizing search fails to maximize social welfare, but is it still worth doing for society?

An obvious measure for this argument is surplus difference between the two economies, with and without search. If a firm chooses intensity $n^* > 0$, then the expected social surplus in this search economy is equal to $W(n^*(\alpha);\alpha)$ in (26). In contrast, without search, social surplus becomes

$$\pi(\underline{\lambda}p_0, \overline{\theta}w_0, w_2) + v_c(\underline{\lambda}p_0) + v_1(\overline{\theta}w_0) + v_2(w_2).$$

In comparison, search is beneficial if the surplus difference is positive, and is harmful if it is negative.

After rearrangement, this is to compare the firm's profit difference

$$\phi(\alpha) - \pi(\underline{\lambda} p_0, \overline{\theta} w_0, w_2) \tag{32}$$

with the total utility differences of consumers and laborers

$$-\int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0) f^*(\lambda_m | n_\lambda)\, d\lambda_m + v_c(\underline{\lambda} p_0) - \int_{\underline{\theta}}^{\overline{\theta}} v_1(\theta_m w_0) g^*(\theta_m | n_\theta)\, d\theta_m + v_1(\overline{\theta} w_0). \tag{33}$$

By the properties of $\pi(\cdot)$ and $v_j(\cdot)$, both (32) and (33) are nonnegative. So, clearly, there are conflicts between a firm and its customers and laborers.

While profit is monetary, utility is a subjective measure. Hence, it is helpful, especially in practice, to reexpress the utility difference in (33) in monetary scale. It is noted, however, that utility theory is purely ordinal in nature and thus all we can do is approximation.

One useful tool for this purpose is Hicks's equivalent variation (EV). To derive it, first rewrite consumer utility and laborer i's utility by using indirect money metric function:

$$e_c(p', v_c(p)) = \min_{\hat{y}, \hat{z}} p' \hat{y} + \hat{z} \quad \text{subject to } u_c(\hat{y}, \hat{z}) = v_c(p)$$

$$e_i(v_i(w_i)) = \min_{\hat{z}, \hat{x}_i} \hat{z} \quad \text{subject to } u_i(\hat{z}, L - \hat{x}_i) = v_i(w_i)$$

where p and p' are two different prices of commodity y.

For consumer utility in economies with and without search, set $p = \lambda_m p_0$ for an economy with search and $p = \underline{\lambda} p_0$ for an economy without search while setting $p' = \underline{\lambda} p_0$ for both economies to get

$$e_c(\underline{\lambda} p_0, v_c(\lambda_m p_0)) = \underline{\lambda} p_0\, \hat{y}(\underline{\lambda} p_0, v_c(\lambda_m p_0)) + \hat{z}(\underline{\lambda} p_0, v_c(\lambda_m p_0))$$

$$e_c(\underline{\lambda} p_0, v_c(\underline{\lambda} p_0)) = \underline{\lambda} p_0\, \hat{y}(\underline{\lambda} p_0, v_c(\underline{\lambda} p_0)) + \hat{z}(\underline{\lambda} p_0, v_c(\underline{\lambda} p_0)).$$

For laborer 1's utility in economies with and without search, set $w = \theta_m w_0$ for an economy with search and $w = \overline{\theta} w_0$ for an economy without search to have

$$e_1(v_1(\theta_m w_0)) = \hat{z}(v_1(\theta_m w_0))$$

$$e_1(v_1(\overline{\theta} w_0)) = \hat{z}(v_1(\overline{\theta} w_0)).$$

The expected EV for consumer, EV_c, is obtained by taking the difference of the above e_c's and then taking the expectation with respect to F^*, so that

$$EV_c = \int_{\underline{\lambda}}^{\overline{\lambda}} \left[\underline{\lambda} p_0\, \hat{y}(\underline{\lambda} p_0, v_c(\lambda_m p_0)) + \hat{z}(\underline{\lambda} p_0, v_c(\lambda_m p_0))\right] f^*(\lambda_m | n_\lambda)\, d\lambda_m$$
$$- \underline{\lambda} p_0\, \hat{y}(\underline{\lambda} p_0, v_c(\underline{\lambda} p_0)) - \hat{z}(\underline{\lambda} p_0, v_c(\underline{\lambda} p_0)).$$

Likewise, the expected EV for laborer 1, EV_1 is derived by taking the difference of the above e_1's and then taking the expectation with respect to G^*, so that

$$\begin{aligned} EV_1 &= \int_{\underline{\theta}}^{\overline{\theta}} \hat{z}(v_1(\theta_m w_0)) g^*(\theta_m | n_\theta)\, d\theta_m - \hat{z}(v_1(\overline{\theta} w_0)) \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \theta_m w_0\, \hat{x}_1(v_1(\theta_m w_0)) g^*(\theta_m | n_\theta)\, d\theta_m - \overline{\theta} w_0\, \hat{x}_1(v_1(\overline{\theta} w_0)), \end{aligned}$$

where the last equality follows from the constraint in the original problem in (25). Applying these, we summarize this discussion as follows.

Proposition 9. *Suppose that (9) holds. Search is worth doing if $\phi(\alpha) - \pi(\underline{\lambda} p_0, \overline{\theta} w_0, w_2) > -EV_c - EV_1$ holds.*

7 Extensions and Limitations

In this paper, we analyzed a simultaneous fixed-sample-size search problem of a competitive firm. To conclude, we remark some extensions and limitations of the model presented in this paper.

The model discussed above is based on several assumptions. While some are essential assumptions to construct a firm's search problem, others are rather for simplifying purposes. For example, multiple-markets setting in Assumptions 2 and 3 are essential, but markets need not be competitive if one condition explained below is satisfied.

Consider a monopolist. He faces a set of consumer markets, each of which has a well-defined, *downward-sloping* inverse demand function $p(\cdot) \in \mathcal{P}$ where $\mathcal{P}$ represents a set of inverse demand functions for commodity y in the economy. Furthermore, as before, he knows the distribution of demand functions he faces (*i.e.*, market types) and the frequencies, but is ignorant of the exact location of any specific function. Let the rest of the setting remain the same.

Begin with stage 2. The monopolist problem in this stage is production problem. That is, given inverse demand $p(\cdot) \in \mathcal{P}$ and input prices θw_0 and w_2, he maximizes direct profit $p(y)y - c(y)$ with respect to y, where $c(y)$ is the firm's cost function for a fixed y such that

$$c(y) = c(\theta w_0, w2, y) = \min_{x_1, x_2} \theta w_0 x_1 + w_2 x_2 \quad \text{subject to } \zeta(x_1, x_2) = y. \tag{34}$$

Let y^* be the monopolist's optimal supply. Then, it necessarily satisfies

$$\frac{\partial p(y)}{\partial y}\, y + p(y) = \frac{\partial c(y)}{\partial y}. \tag{35}$$

The argument minimum in (34) at y^* gives the monopolist's input demands x_i^*. Substituting these into the direct profit yields indirect profit π.

Go to stage 1. By conducting search of intensity n, a monopolist observes n_λ inverse demand functions, $p_1(\cdot), p_2(\cdot), \ldots, p_{n_\lambda}(\cdot)$, and n_θ discount factors, $\theta_1, \theta_2, \ldots, \theta_{n_\theta}$. Let i and j be integers such that $1 \leq i \leq n_\lambda$ and $1 \leq j \leq n_\theta$. For each $(p_i(\cdot), \theta_j)$, he considers a maximization problem such that

$$\max_y \; p_i(y)y - c(\theta_j w_0, w_2, y). \tag{36}$$

Let y_{ij}^* be the profit-maximizing quantity supplied for this $(p_i(\cdot), \theta_j)$-pair and let $p_{ij} = p_i(y_{ij}^*)$ be a selling price associated with this quantity. The monopolist's indirect profit π for this pair $(p_i(\cdot), \theta_j)$ is then

$$\pi_{i\,j} = p_{ij} y_{ij}^* - c(\theta_j w_0, w_2, y_{ij}^*)$$

In case of n intensity of search, the monopolist obtains in total $n_\lambda n_\theta$ indirect profits to compare.

Before going to search, he can compute y_{ij}^* for each possible pair of $(p_i(\cdot), \theta_j)$. Fix θ_j and consider a set $\mathcal{P}' = \{p_{ij} \in \mathbb{R}_+ : i \text{ is such that } p_i(\cdot) \in \mathcal{P}\}$. Since $p_{ij} \in \mathcal{P}'$ is a real number, we can place $p_i(\cdot)$ in ascending order to construct an increasing sequence $\langle p_{(s)}(\cdot)\rangle = \langle p_{(1)}(\cdot), p_{(2)}(\cdot), \ldots, p_{(n_\lambda)}(\cdot)\rangle$ for the fixed θ_j. *If ordering of $p_i(\cdot)$ in $\langle p_{(s)}(\cdot)\rangle$ does not change with θ_j*, then this s successfully captures the profitability of demand functions $p(\cdot) \in \mathcal{P}$, since π is nondecreasing in selling price. That is, in that case, if we let

$$\pi_{(s)\,j} = p_{(s)}(y_{ij}^*) y_{ij}^* - c(y_{ij}^*),$$

then s satisfies $\pi_{(s+1)\,j} \geq \pi_{(s),j}$ for any j.

If the demand functions $p(\cdot) \in \mathcal{P}$ can be ordered according to profitability, then it holds that for any intensity n

$$\pi^* \geq \pi_{i\,j} \quad \text{for any } i \text{ and } j,$$

where π^* is indirect profit when the monopolist faces inverse demand function $p_{(n_\lambda)}(\cdot)$ and discount factor θ_m. Therefore, it is clear that, out of n_λ inverse demand functions, $p_1(\cdot), p_2(\cdot), \ldots, p_{n_\lambda}(\cdot)$, and n_θ discount factors, $\theta_1, \theta_2, \ldots, \theta_{n_\theta}$, he chooses $p_{(n_\lambda)}$ and θ_m. In other words, the monopolist simply seeks the most profitable consumer market for commodity y and the lowest input price for input x_1. The monopolist's objective function in his search problem then becomes analogous to (5).

The key that enables us to use the simultaneous search model of a competitive firm to a monopolist's search is whether we can order different demand functions faced by a monopolist according to profitability. If this condition fails, then our formulation presented is inappropriate since the firm's expected profit function can no longer be written by using pdf's f^* and g^* like before. If that condition still holds, then this approach remains useful in studying the search behavior of monopolists and monopsonists.

Appendix 1

This appendix contains the proofs of Propositions 2, 3 and 6–8 in the text.

Proof of Proposition 2. The second-order sufficient condition for a relative maximum is that the Hessian of H, D^2H, is negative definite in some neighborhood of n^*. The determinantal test for it is that every k^{th} leading principal minor of $|D^2H|$ is positive if k is even, and negative otherwise. D^2H is

$$\begin{bmatrix} H_{n_\lambda n_\lambda} & H_{n_\lambda n_\theta} \\ H_{n_\theta n_\lambda} & H_{n_\theta n_\theta} \end{bmatrix},$$

where

$$H_{n_\lambda n_\lambda} = \frac{\partial^2 E(\pi|n_\lambda, n_\theta)}{\partial {n_\lambda}^2} = \int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\cdot) \frac{\partial^2 f^*(\lambda_m|n_\lambda)}{\partial {n_\lambda}^2} g^*(\theta_m|n_\theta)\, d\lambda_m\, d\theta_m, \tag{37}$$

$$H_{n_\lambda n_\theta} = H_{n_\theta n_\lambda} = \frac{\partial^2 E(\pi|n_\lambda, n_\theta)}{\partial n_\lambda \partial n_\theta} = \int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\cdot) \frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda} \frac{\partial g^*(\theta_m|n_\theta)}{\partial n_\theta}\, d\lambda_m\, d\theta_m, \tag{38}$$

and

$$H_{n_\theta n_\theta} = \frac{\partial^2 E(\pi|n_\lambda, n_\theta)}{\partial {n_\theta}^2} = \int_{\underline{\lambda}}^{\overline{\lambda}} \int_{\underline{\theta}}^{\overline{\theta}} \pi(\cdot) f^*(\lambda_m|n_\lambda) \frac{\partial^2 g^*(\theta_m|n_\theta)}{\partial {n_\theta}^2}\, d\lambda_m\, d\theta_m. \tag{39}$$

The first leading principal minors are (37) and (39). The determinantal test insists that these be positive for $n = n^*$. Corollaries 5 and 6 in Appendix 2 imply that both of them hold, since $\pi(\cdot)$ is increasing in $\lambda_m p_0$ and decreasing in $\theta_m w_0$, given Assumption 8.

The second principal minor is $|D^2H| = H_{n_\lambda n_\lambda} H_{n_\theta n_\theta} - [H_{n_\lambda n_\theta}]^2$. The above test insists that it be positive for $n = n^*$. (9) is just this rearranged.

Global optimality follows if those respective conditions hold for all n. □

Proof of Proposition 3. Here, we use the approach developed by Silberberg (1974). The firm's (original) problem is $\max_n H(n; \alpha)$ as in (5). Here, $n = (n_\lambda, n_\theta)$ and $\alpha = (c_\lambda, c_\theta, p_0, w_0, w_2)$. The "primal-dual" problem of this maximization problem is $\min_{n,a} \phi(\alpha) - H(n; \alpha)$. The Lagrangean of this primal-dual problem is then

$$L(n, \alpha) = \phi(\alpha) - H(n; \alpha). \tag{40}$$

By construction, $L(n^*,\alpha)$ necessarily satisfies the first- and second-order necessary conditions for a *minimum*. They are

$$DL(n^*,\alpha)=\begin{bmatrix}L_n(n^*,\alpha) & L_\alpha(n^*,\alpha)\end{bmatrix}=\begin{bmatrix}H_n(n^*;\alpha) & \phi_\alpha(\alpha)-H_\alpha(n^*,\alpha)\end{bmatrix}\equiv 0 \tag{41}$$

$$D^2L(n^*,\alpha)=\begin{bmatrix}L_{nn}(n^*,\alpha) & L_{n\alpha}(n^*,\alpha)\\ L_{\alpha n}(n^*,\alpha) & L_{\alpha\alpha}(n^*,\alpha)\end{bmatrix}\text{ is positive semidefinite.} \tag{42}$$

The latter condition (42) implies that its submatrix $L_{\alpha\alpha}(n^*,\alpha)$ also satisfies the positive semidefiniteness.

Twice differentiating the both sides of (40) with respect to α gives

$$L_{\alpha\alpha}(n,\alpha)=\phi_{\alpha\alpha}(\alpha)-H_{\alpha\alpha}(n,\alpha).$$

But at a neighborhood of n^*, (41) ensures $\phi_\alpha(\alpha)\equiv H_\alpha(n^*,\alpha)$. Differentiate its both sides with respect to α, yielding

$$\phi_{\alpha\alpha}=H_{\alpha n}(n^*,\alpha)\cdot\frac{\partial n^*}{\partial\alpha}+H_{\alpha\alpha}.$$

By substitution, we obtain

$$L_{\alpha\alpha}=H_{\alpha n}(n^*,\alpha)\cdot\frac{\partial n^*}{\partial\alpha}.$$

From

$$H_\alpha=\begin{bmatrix}H_{c_\lambda} & H_{c_\theta} & H_{p_0} & H_{w_0} & H_{w_2}\end{bmatrix}=\begin{bmatrix}-n_\lambda & -n_\theta & \frac{\partial E(\pi|n)}{\partial(\lambda_m p_0)}\lambda_m & \frac{\partial E(\pi|n)}{\partial(\theta_m w_0)}\theta_m & \frac{\partial E(\pi|n)}{\partial w_2}\end{bmatrix},$$

each element of $H_{\alpha n}$ are calculated as

$$\begin{aligned}
H_{c_\lambda n_\lambda}&=-1,\\
H_{c_\lambda n_\theta}&=0,\\
H_{c_\theta n_\lambda}&=0,\\
H_{c_\theta n_\theta}&=-1,\\
H_{p_0 n_\lambda}&=\frac{\partial^2E(\pi|n)}{\partial(\lambda_m p_0)\,\partial n_\lambda}\lambda_m>0,\\
H_{p_0 n_\theta}&=\frac{\partial^2E(\pi|n)}{\partial(\lambda_m p_0)\,\partial n_\theta}\lambda_m=0,\\
H_{w_0 n_\lambda}&=\frac{\partial^2E(\pi|n)}{\partial(\theta_m w_0)\,\partial n_\lambda}\theta_m=0,\\
H_{w_0 n_\theta}&=\frac{\partial^2E(\pi|n)}{\partial(\theta_m w_0)\,\partial n_\theta}\theta_m>0,\\
H_{w_2 n_\lambda}&=\frac{\partial^2E(\pi|n)}{\partial w_2\,\partial n_\lambda}=0,
\end{aligned}$$

$$H_{w_2 n_\theta} = \frac{\partial^2 E(\pi|n)}{\partial w_2 \, \partial n_\theta} = 0.$$

Here, the signs of the last six elements follow from the theorems and corollaries in Appendix 2 and the Envelop Theorem. In matrix notation,

$$H_{\alpha n} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ H_{p_0 n_\lambda} & 0 \\ 0 & H_{w_0 n_\theta} \\ 0 & 0 \end{bmatrix}.$$

So,

$$L_{\alpha\alpha} = \begin{bmatrix} -\frac{\partial n_\lambda^*}{\partial c_\lambda} & -\frac{\partial n_\lambda^*}{\partial c_\theta} & -\frac{\partial n_\lambda^*}{\partial p_0} & -\frac{\partial n_\lambda^*}{\partial w_0} & -\frac{\partial n_\lambda^*}{\partial w_2} \\ -\frac{\partial n_\theta^*}{\partial c_\lambda} & -\frac{\partial n_\theta^*}{\partial c_\theta} & -\frac{\partial n_\theta^*}{\partial p_0} & -\frac{\partial n_\theta^*}{\partial w_0} & -\frac{\partial n_\theta^*}{\partial w_2} \\ H_{p_0 n_\lambda}\frac{\partial n_\lambda^*}{\partial c_\lambda} & H_{p_0 n_\lambda}\frac{\partial n_\lambda^*}{\partial c_\theta} & H_{p_0 n_\lambda}\frac{\partial n_\lambda^*}{\partial p_0} & H_{p_0 n_\lambda}\frac{\partial n_\lambda^*}{\partial w_0} & H_{p_0 n_\lambda}\frac{\partial n_\lambda^*}{\partial w_2} \\ H_{w_0 n_\theta}\frac{\partial n_\theta^*}{\partial c_\lambda} & H_{w_0 n_\theta}\frac{\partial n_\theta^*}{\partial c_\theta} & H_{w_0 n_\theta}\frac{\partial n_\theta^*}{\partial p_0} & H_{w_0 n_\theta}\frac{\partial n_\theta^*}{\partial w_0} & H_{w_0 n_\theta}\frac{\partial n_\theta^*}{\partial w_2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{43}$$

To establish parts (i) and (iv) of the proposition, note that the positive semidefiniteness of $L_{\alpha\alpha}^*$ implies that its first leading principal minors are nonnegative. That is,

$$\begin{aligned} -\frac{\partial n_\lambda^*}{\partial c_\lambda} \geq 0 \quad &\Longrightarrow \quad \frac{\partial n_\lambda^*}{\partial c_\lambda} \leq 0, \\ -\frac{\partial n_\theta^*}{\partial c_\theta} \geq 0 \quad &\Longrightarrow \quad \frac{\partial n_\theta^*}{\partial c_\theta} \leq 0, \\ H_{p_0 n_\lambda}\frac{\partial n_\lambda^*}{\partial p_0} \geq 0 \quad &\Longrightarrow \quad \frac{\partial n_\lambda^*}{\partial p_0} \geq 0, \\ H_{w_0 n_\theta}\frac{\partial n_\theta^*}{\partial w_0} \geq 0 \quad &\Longrightarrow \quad \frac{\partial n_\theta^*}{\partial w_0} \geq 0. \end{aligned}$$

The last two inequalities are parts (i) and (iv) of the proposition.

By Young Theorem, $D^2 L$ is symmetric. And so is $L_{\alpha\alpha}$. The symmetry of $L_{\alpha\alpha}^*$ implies

$$-\frac{\partial n_\lambda^*}{\partial c_\theta} = -\frac{\partial n_\theta^*}{\partial c_\lambda} \quad \Longrightarrow \quad \frac{\partial n_\lambda^*}{\partial c_\theta} = \frac{\partial n_\theta^*}{\partial c_\lambda}, \tag{44}$$

$$-\frac{\partial n_\lambda^*}{\partial w_0} = H_{w_0 n_\theta}\frac{\partial n_\theta^*}{\partial c_\lambda} \quad \Longrightarrow \quad \mathrm{sign}(\frac{\partial n_\lambda^*}{\partial w_0}) = \mathrm{sign}(-\frac{\partial n_\theta^*}{\partial c_\lambda}), \tag{45}$$

$$-\frac{\partial n_\theta^*}{\partial p_0} = H_{p_0 n_\lambda}\frac{\partial n_\lambda^*}{\partial c_\theta} \quad \Longrightarrow \quad \mathrm{sign}(\frac{\partial n_\theta^*}{\partial p_0}) = \mathrm{sign}(-\frac{\partial n_\lambda^*}{\partial c_\theta}), \tag{46}$$

$$\frac{\partial n_\lambda^*}{\partial w_2} = 0,$$

$$\frac{\partial n_\theta^*}{\partial w_2} = 0.$$

The last two equations are parts (v) and (vi).

For the rest of the statement, we need look at entire $D^2L = \begin{bmatrix} L_{nn} & L_{n\alpha} \\ L_{\alpha n} & L_{\alpha\alpha} \end{bmatrix}$, where

$$L^*_{nn} = \begin{bmatrix} -H_{n_\lambda n_\lambda} & -H_{n_\lambda n_\theta} \\ -H_{n_\theta n_\lambda} & -H_{n_\theta n_\theta} \end{bmatrix},$$

$$L^*_{n\alpha} = \begin{bmatrix} -H_{n_\lambda c_\lambda} & -H_{n_\lambda c_\lambda} & -H_{n_\lambda p_0} & -H_{n_\lambda w_0} & -H_{n_\lambda w_2} \\ -H_{n_\theta c_\lambda} & -H_{n_\theta c_\lambda} & -H_{n_\theta p_0} & -H_{n_\theta w_0} & -H_{n_\theta w_2} \end{bmatrix} = L^{*\ \top}_{\alpha n}.$$

By direct calculation, D^2L is written as

$$\begin{bmatrix} -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda^2} & -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda \partial n_\theta} & 1 & 0 & -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda \partial(\lambda_m p_0)}\lambda_m & -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda \partial(\theta_m w_0)}\theta_m & -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda \partial w_2} \\ -\frac{\partial^2 E(\pi|n)}{\partial n_\theta \partial n_\lambda} & -\frac{\partial^2 E(\pi|n)}{\partial n_\theta^2} & 0 & 1 & -\frac{\partial^2 E(\pi|n)}{\partial n_\theta \partial(\lambda_m p_0)}\lambda_m & -\frac{\partial^2 E(\pi|n)}{\partial n_\theta \partial(\theta_m w_0)}\theta_m & -\frac{\partial^2 E(\pi|n)}{\partial n_\theta \partial w_2} \\ 1 & 0 & -\frac{\partial n_\lambda^*}{\partial c_\lambda} & -\frac{\partial n_\lambda^*}{\partial c_\theta} & 0 & 0 & 0 \\ 0 & 1 & -\frac{\partial n_\theta^*}{\partial c_\lambda} & -\frac{\partial n_\theta^*}{\partial c_\theta} & 0 & 0 & 0 \\ -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda \partial(\lambda_m p_0)}\lambda_m & -\frac{\partial^2 E(\pi|n)}{\partial n_\theta \partial(\lambda_m p_0)}\lambda_m & 0 & 0 & 0 & 0 & 0 \\ -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda \partial(\theta_m w_0)}\theta_m & -\frac{\partial^2 E(\pi|n)}{\partial n_\theta \partial(\theta_m w_0)}\theta_m & 0 & 0 & 0 & 0 & 0 \\ -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda \partial w_2} & -\frac{\partial^2 E(\pi|n)}{\partial n_\theta \partial w_2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The positive semidefiniteness of $D^2L^*(n^*, \alpha)$ ensures that all of its principal minors be nonnegative. In particular, this implies that the following principal minors M_2' and M_2'' are nonnegative:

$$M_2' = \begin{vmatrix} -\frac{\partial^2 E(\pi|n)}{\partial n_\lambda^2} & 1 \\ 1 & -\frac{\partial n_\lambda^*}{\partial c_\lambda} \end{vmatrix} \geq 0 \quad \Longrightarrow \quad \frac{\partial n_\lambda^*}{\partial c_\lambda} \leq \frac{1}{\frac{\partial^2 E(\pi|n)}{\partial n_\lambda^2}}, \tag{47}$$

$$M_2'' = \begin{vmatrix} -\frac{\partial^2 E(\pi|n)}{\partial n_\theta^2} & 1 \\ 1 & -\frac{\partial n_\theta^*}{\partial c_\theta} \end{vmatrix} \geq 0 \quad \Longrightarrow \quad \frac{\partial n_\theta^*}{\partial c_\theta} \leq \frac{1}{\frac{\partial^2 E(\pi|n)}{\partial n_\theta^2}} < 0$$

where M_2' consists of D^2L's first and third rows and columns while M_2'' consists of its second and fourth rows and columns. Because π is increasing in $\lambda_m p_0$ under Assumption 8, Corollary 5 in Appendix 2 insists $\frac{\partial^2 E(\pi|n)}{\partial n_\lambda^2} < 0$. Applying this to the above inequalities establishes parts (vii) and (viii).

Lastly, to show parts (ii), (iii), (ix) and (x), consider the following third principal mninor M_3

$$M_3 = \begin{vmatrix} -\frac{\partial^2 E[\pi|n]}{\partial n_\lambda^2} & 1 & 0 \\ 1 & -\frac{\partial n_\lambda^*}{\partial c_\lambda} & -\frac{\partial n_\lambda^*}{\partial c_\theta} \\ 0 & -\frac{\partial n_\theta^*}{\partial c_\lambda} & -\frac{\partial n_\theta^*}{\partial c_\theta} \end{vmatrix},$$

which consists of D^2L's first, third and fourth rows and columns. The nonnegativity of M_3 implies that

$$\frac{\partial n_\theta^*}{\partial c_\theta} \geq \left[\frac{\partial n_\lambda^*}{\partial c_\lambda}\frac{\partial n_\theta^*}{\partial c_\theta} - \frac{\partial n_\lambda^*}{\partial c_\theta}\frac{\partial n_\theta^*}{\partial c_\lambda} \right] \frac{\partial^2 E(\pi|n)}{\partial n_\lambda^2}.$$

Rearranging the terms and then applying (47) gives

$$\left[\frac{\partial n_\lambda^*}{\partial c_\lambda}\frac{\partial n_\theta^*}{\partial c_\theta} - \frac{\partial n_\lambda^*}{\partial c_\theta}\frac{\partial n_\theta^*}{\partial c_\lambda}\right] \geq \frac{\partial n_\theta^*}{\partial c_\theta}\frac{1}{\frac{\partial^2 E(\pi|n)}{\partial n_\lambda^2}} \geq \frac{\partial n_\theta^*}{\partial c_\theta}\frac{\partial n_\lambda^*}{\partial c_\lambda}.$$

Together with (44), this proves parts (ix) and (x). Together with (45) and (46), this in turn establishes parts (ii) and (iii). □

Proof of Proposition 6. The second-order sufficient condition for a local (global) maximum is that the Hessian of W, D^2W, is negative definite in some neighborhood of n^{**} (for all $n \in \mathbb{R}_+^2$ respectively). The determinantal test for this condition is that every k^{th} leading principal minor of $|D^2W|$ is positive if k is even, and negative otherwise. (29)-(31) just restates these requirements. □

Proof of Proposition 7. It suffices to show that $n^* \neq n^{**}$. By Propositions 1 and 5, $n_\lambda^* = n_\lambda^{**}$ holds if and only if $H_{n_\lambda} = W_{n_\lambda}$ holds. From (7) and (27), the latter holds if and only if

$$\int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0)\frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda}\, d\lambda_m = 0.$$

Corollary 3 implies that this is the case if and only if v_c is constant over p, or $\frac{\partial v_c(\cdot)}{\partial(\lambda_m p_0)} = 0$, since v_c is nonincreasing in $\lambda_m p_0$. However, it is not possible under the stated assumptions, since together with Roy's identity $\frac{\partial v_c(\cdot)}{\partial(\lambda_m p_0)} = 0$ implies $y^* = 0$ but Assumption 8 ensures $y^* > 0$. The other claim, $n_\theta^* \neq n_\theta^{**}$, can be established analogously. □

Proof of Proposition 8. Under the stated conditions, n^* and n^{**} exist. We show $n_{\lambda_m}^* > n_{\lambda_m}^{**}$. The proof of the other claim is similar. First, given Assumption 8, π increases in $\lambda_m p_0$. So, Corollary 5 implies that H_{n_λ} decreases in n_λ. Given (29), W_{n_λ} decreases in n_λ as well. From Corollary 1 together with the above proof of Proposition 7, we have

$$\int_{\underline{\lambda}}^{\overline{\lambda}} v_c(\lambda_m p_0)\frac{\partial f^*(\lambda_m|n_\lambda)}{\partial n_\lambda}\, d\lambda_m > 0.$$

So, $H_{n_\lambda} > W n_\lambda$ holds for any n_λ. Both H_{n_λ} and W_{n_λ} are monotonic, and hence invertible. Hence, $H_{n_\lambda}^{-1}(0) = n_\lambda^* > n_\lambda^{**} = W_{n_\lambda}^{-1}(0)$ follows. □

Appendix 2

In this appendix, we present six corollaries of Theorem 1 and 2 by Manning and Morgan (1982), all of which are repeatedly referred to in the proof of the propositions as well as in the text.

Let f^* be the pdf of the maximum of a sample of n_x independent and identically distributed observations from a population with pdf f and cdf F. Likewise, denote by g^* the pdf of the maximum of a sample of n_y independent and identically distributed observations from a population with pdf g and cdf G. Let us assume that both of the support of f and g are reals $\mathbb{R}$. Lastly, h is a differentiable function.

First, we replicate Theorem 1 and 2 by Manning and Morgan (1982) for reference.

Theorem 1.

$$\int_{\mathbb{R}} h(y)\frac{\partial g^*(y|n_y)}{\partial n_y}\,dy \gtreqless 0, \ \text{ as } \frac{dh}{dy} \lesseqgtr 0, \ \text{ for all } y \in \mathbb{R}. \tag{48}$$

Theorem 2.

$$\int_{\mathbb{R}} h(y)\frac{\partial^2 g^*(y|n_y)}{\partial {n_y}^2}\,dy \gtreqless 0, \ \text{ as } \frac{dh}{dy} \gtreqless 0, \ \text{ for all } y \in \mathbb{R}.$$

The next two corollaries are maximum counterpart of these theorems.

Corollary 1.

$$\int_{\mathbb{R}} h(x)\frac{\partial f^*(x|n_x)}{\partial n_x}\,dx \gtreqless 0, \ \text{ as } \frac{dh}{dx} \gtreqless 0, \ \text{ for all } x \in \mathbb{R}.$$

Proof. From (3),

$$\frac{\partial f^*(x|n_x)}{\partial n_x} = [1 + n_x \ln F(x)]F(x)^{n_x - 1} f(x). \tag{49}$$

There exists a unique $x = r$ such that

$$1 + n_x \ln F(x) \lesseqgtr 0 \quad \text{as } x \lesseqgtr r,$$

since f is a pdf. Thus,

$$\frac{\partial f^*(x|n_x)}{\partial n_x} \lesseqgtr 0 \quad \text{as } x \lesseqgtr r.$$

Suppose that $\frac{dh}{dx} > 0$ for all $x \in \mathbb{R}$. Then $h(x) \lesseqgtr h(r)$ when $x \lesseqgtr r$, and

$$\begin{aligned}
\int_{\mathbb{R}} h(x)\frac{\partial f^*(x|n_x)}{\partial n_x}\,dx &= \int^{r} h(x)\frac{\partial f^*(x|n_x)}{\partial n_x}\,dx + \int_{r} h(x)\frac{\partial f^*(x|n_x)}{\partial n_x}\,dx \\
&> \int^{r} h(r)\frac{\partial f^*(x|n_x)}{\partial n_x}\,dx + \int_{r} h(r,y)\frac{\partial f^*(x|n_x)}{\partial n_x}\,dx \\
&= h(r)\int_{\mathbb{R}} \frac{\partial f^*(x|n_x)}{\partial n_x}\,dx = 0
\end{aligned}$$

since f^* is a pdf. A similar argument applies if $\frac{dh}{dx} \leq 0$. □

Corollary 2.

$$\int_{\mathbb{R}^2} h(x) \frac{\partial^2 f^*(x|n_x)}{\partial {n_x}^2}\, dx \gtreqless 0, \ \text{ as } \ \frac{dh}{dx} \lesseqgtr 0, \ \text{ for all } x \in \mathbb{R}.$$

Proof. From (49),

$$\frac{\partial^2 f^*(x|n_x)}{\partial {n_x}^2} = [2 + n_x \ln F(x)] F(x)^{n_x - 1} \ln F(x) f(x).$$

Since f is a pdf, there is a unique $x = r'$ such that

$$2 + n_x \ln F(x) \gtreqless 0 \quad \text{as } x \lesseqgtr r'.$$

Thus,

$$\frac{\partial f^*(x|n_x)}{\partial n_x} \lesseqgtr 0 \quad \text{as } x \lesseqgtr r'.$$

Proceeding as in Corollary 1 completes the proof. □

From the above four theorems and corollaries, the next statements are immediate.

Corollary 3.

$$\iint_{\mathbb{R}^2} h(x, y) \frac{\partial f^*(x|n_x)}{\partial n_x} g^*(y|n_y)\, dx\, dy \gtreqless 0, \ \text{ as } \ \frac{\partial h}{\partial x} \gtreqless 0, \ \text{ for all } (x, y) \in \mathbb{R}^2.$$

Proof. Corollary 1 implies that, fixing $y \in \mathbb{R}$,

$$\int_{\mathbb{R}} h(x, y) \frac{\partial f^*(y|n_x}{\partial n_x}\, dx \gtreqless 0, \ \text{ as } \ \frac{\partial h}{\partial x} \gtreqless 0, \ \text{ for all } (x, y) \in \mathbb{R}^2.$$

Integrate over y, resulting the claim. □

Corollary 4.

$$\iint_{\mathbb{R}^2} h(x, y) f^*(x|n_x) \frac{\partial g^*(y|n_y)}{\partial n_y}\, dx\, dy \gtreqless 0, \ \text{ as } \ \frac{\partial h}{\partial y} \lesseqgtr 0, \ \text{ for all } (x, y) \in \mathbb{R}^2.$$

Proof. Theorem 1 implies that, fixing $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} h(x, y) \frac{\partial g^*(y|n_y)}{\partial n_y}\, dy \gtreqless 0, \ \text{ as } \ \frac{\partial h}{\partial y} \lesseqgtr 0, \ \text{ for all } y \in \mathbb{R}.$$

Integrate over x, resulting the claim. □

Corollary 5.

$$\iint_{\mathbb{R}^2} h(x, y) \frac{\partial^2 f^*(x|n_x)}{\partial {n_x}^2} g^*(y|n_y)\, dx\, dy \gtreqless 0, \ \text{ as } \ \frac{\partial h}{\partial x} \lesseqgtr 0, \ \text{ for all } (x, y) \in \mathbb{R}^2.$$

Proof. Corollary 2 implies that, fixing $y \in \mathbb{R}$,

$$\int_{\mathbb{R}} h(x,y) \frac{\partial^2 f^*(y|n_x}{\partial {n_x}^2}\, dx \gtreqless 0, \text{ as } \frac{\partial h}{\partial x} \lesseqgtr 0, \text{ for all } (x,y) \in \mathbb{R}^2.$$

Integrate over y, resulting the claim. □

Corollary 6.

$$\iint_{\mathbb{R}^2} h(x,y) f^*(x|n_x) \frac{\partial^2 g^*(y|n_y)}{\partial {n_y}^2}\, dx\, dy \gtreqless 0, \text{ as } \frac{\partial h}{\partial y} \gtreqless 0, \text{ for all } (x,y) \in \mathbb{R}^2.$$

Proof. Theorem 2 implies that, fixing $x \in \mathbb{R}$

$$\int_{\mathbb{R}} h(x,y) \frac{\partial^2 g^*(y|n_y)}{\partial {n_y}^2}\, dy \gtreqless 0, \text{ as } \frac{\partial h}{\partial y} \gtreqless 0, \text{ for all } y \in \mathbb{R}.$$

Integrate over x, resulting the claim. □

References

Chade, Hector and Lones Smith (2006) “Simultaneous Search,” *Econometrica*, Vol. 74, No. 5, pp. 1293–1307.

Manning, Richard and Peter B. Morgan (1982) “Search and Consumer Theory,” *Review of Economic Studies*, Vol. 49, No. 2, pp. 203–216.

Morgan, Peter B. (1983) “Search and Optimal Sample Sizes,” *Review of Economic Studies*, Vol. 50, pp. 659–675.

Morgan, Peter B. and Richard Manning (1985) “Optimal Search,” *Econometrica*, Vol. 53, No. 4, pp. 923–944.

Rothschild, Michael (1974) “Search for the Lowest Price When the Distribution of Prices Is Not Known,” *Journal of Political Economy*, Vol. 82, No. 4, pp. 689–711.

Silberberg, Eugene (1974) “A Revision of Comparative Statics Methodology in Economics, or, How to Do Comparative Statics on the Back of an Envelope,” *Journal of Economic Theory*, Vol. 7, No. 2, pp. 159–172.

Stigler, George J. (1961) “The Economics of Information,” *Journal of Political Economy*, Vol. 69, No. 3, pp. 213–225.

Weitzman, Martin L. (1971) “Optimal Search for the Best Alternative,” *Econometrica*, Vol. 47, No. 3, pp. 641–654.