Bonner's Solution and Double Reissner-Nordström Solution

AZUMA Takahiro

ボナー解と二重ライスナー・ノルドシュトルム解

東　孝博

Abstract

Bonner's solution, one of the static two-body solutions of Einstein-Maxwell equations, is investigated in comparison with the $N = 2$ case of the axisymmetric N-Reissner-Nordström solution. The physical masses and electric charges of each body are calculated by using the Komar integrals. The results obtained from these calculations differ from those given by Bonner. The equilibrium condition is also studied, which shows that Bonner's solution reduces to the one-body type of Majumder-Papapetreu solution under the equilibrium condition, while the case of the Reisser-Nordström solution to the two-body type.

要旨

アインシュタイン・マクスウェル方程式の静的２体解のひとつであるW.B.ボナーによる解を、軸対称 N 体ライスナー・ノルドシュトルム解の $N = 2$ の場合との比較で調べる。それぞれの重力源の物理量である電荷と質量をコマー積分により求めるが、ボナーによる結果とは異なる。両方の解において２体の平衡条件を課した場合、ライスナー・ノルドシュトルム解が２体のマジャンダー・パパペトロウ解に帰着するのに対して、ボナー解は１体の解に帰着することが分かる。

1 Introduction

Bonner's solution [1], which was presented by W.B. Bonner in 1979, is an exact static and axisymmetric solution of Einstein-Maxwell equations. The solution is not of Weyl class, where the gravitational and electrostatic potentials are not functionally related. Bonner interpreted this solution as describing the gravitational and electric fields created by a mass with an electric charge and dipole moment. Since then, the solution has been reexamined by other workers [2, 3], and they revealed that Bonner's solution describes the fields by two charged masses. In addition, it is shown that the solution has some strange properties that are unknown in classical electrostatics [4].

The two-body system, such as the fields described by Bonner's solution, attracts interest because we can consider the static equilibrium where Coulomb force in electromagnetic field balances gravitational one. Many works [3, 5–14] have been conducted to study the equilibrium condition. In order to elucidate the equilibrium condition imposed on the masses and electric charges of each body, we need to define the individual masses and charges in two-body system. Therefore, the definitions of mass and charge are important in studying the equilibrium condition.

The present author and T.Koikawa [8] presented an exact static and axisymmetric solution of Einstein-Maxwell equations by applying the inverse scattering method that is one of the soliton techniques to solve gravitational field equations. The solution, which is of Weyl class, describes n charged masses located along the symmetry axis; therefore we called it the axisymmetric N-Reissner-Nordström solution. We defined the masses and charges of each body and studied the condition that should be imposed on them for static equilibrium.

In this paper, we investigate Bonner's solution in comparison with the $N = 2$ case of the axisymmetric N-Reissner-Nordström solution. In Bonner's solution, we define the masses and electric charges of each body by means of the Komar integrals [15]. The obtained quantities are not the same as those in the original definitions of Bonner [16]. The equilibrium condition that the symmetry axis between two bodies should be locally Euclidean leads to the result that both the mass and charge of one body disappear. The solution becomes of Weyl class and reduces to the one-body type of Majumder-Papapetrou solution [17, 18]. To compare Bonner's solution with the $N = 2$ case of the axisymmetric N-Reissner-Nordström solution, which also describes the fields by two charged masses, we define the masses and charges of each body and consider the equilibrium condition in this case. The solution becomes the two-body type of Majumdar-Papapetrou solution in equilibrium.

In section 2, firstly, we show that Bonner's solution can be derived from the 2-soliton solution that is obtained by applying the inverse scattering method to Einstein-Maxwell equations. Secondly, we study the structure of solution. Finally, we consider the definitions of charge and mass and the equilibrium condition. In section 3, we present the $N = 2$ case of the axisymmetric N-Reissner-Nordström solution. The structures of solution, the definitions of charge and mass, and the equilibrium condition are compared with those in Bonner's solution. We also present the singularity structure of the solution that satisfies the equilibrium condition. In section 4, we give a brief discussion on the definitions of individual masses and charges.

2 Bonner's solution

2.1 Derivation of solution

Bonner's solution is expressed in the prolate spheroidal coordinates (x, y) as

$$\begin{aligned} ds^2 &= \sigma^2 U^2 V^2 \left[\frac{(UV - BV - CU)^2}{(x^2 - y^2)^3} \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) \right. \\ &+ \left. \frac{(x^2 - 1)(1 - y^2)}{(UV - BV - CU)^2} d\phi^2 \right] - \frac{(UV - BV - CU)^2}{U^2 V^2} dt^2, \quad (2.1.1) \end{aligned}$$

where

$$U = B + Ay - x, \quad V = C + Ay + x. \quad (2.1.2)$$

In these equations, A, B, C and σ are constants satisfying the relation

$$A^2 = BC + 1. \quad (2.1.3)$$

In order to obtain this Bonner's solution by applying the inverse scattering method to Einstein-Maxwell equations, we first consider the metric in the canonical cylindrical coordinates (ρ, z)

$$ds^2 = f^{-1}[Q(d\rho^2 + dz^2) + \rho^2 d\phi^2] - f dt^2, \quad (2.1.4)$$

where f and Q are functions of ρ and z. In the metric (2.1.4), the source-free Einstein-Maxwell equations

$$R^{\mu}{}_{\nu} = 2\left(F^{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta^{\mu}{}_{\nu}F^{\alpha\beta}F_{\alpha\beta}\right), \tag{2.1.5}$$

$$F^{\mu\nu}{}_{;\mu} = 0, \tag{2.1.6}$$

$$F_{\mu\nu} = A_{\nu},_{\mu} - A_{\mu},_{\nu} \tag{2.1.7}$$

are written as

$$(\ln f),_{\rho\rho} + \rho^{-1}(\ln f),_{\rho} + (\ln f),_{zz} = 2f^{-1}(\chi,_{\rho}^{2} + \chi,_{z}^{2}), \tag{2.1.8}$$

$$(\ln Q),_{\rho} = \frac{\rho}{2}[(\ln f),_{\rho}^{2} - (\ln f),_{z}^{2}] - 2\rho f^{-1}(\chi,_{\rho}^{2} - \chi,_{z}^{2}), \tag{2.1.9}$$

$$(\ln Q),_{z} = \rho(\ln f),_{\rho}(\ln f),_{z} - 4\rho f^{-1}\chi,_{\rho}\chi,_{z}, \tag{2.1.10}$$

$$\chi,_{\rho\rho} + \rho^{-1}\chi,_{\rho} + \chi,_{zz} = \chi,_{\rho}(\ln f),_{\rho} + \chi,_{z}(\ln f),_{z}, \tag{2.1.11}$$

where $\chi = -A_0$. We next introduce a function ψ by

$$\begin{cases} \psi,_{z} = -\rho f^{-1}\chi,_{\rho} \\ \psi,_{\rho} = \rho f^{-1}\chi,_{z}, \end{cases} \tag{2.1.12}$$

and rewrite Eqs.(2.1.8)-(2.1.11) as

$$(\ln f),_{\rho\rho} + \rho^{-1}(\ln f),_{\rho} + (\ln f),_{zz} = 2\rho^{-2}f(\psi,_{\rho}^{2} + \psi,_{z}^{2}), \tag{2.1.13}$$

$$(\ln Q),_{\rho} = \frac{\rho}{2}[(\ln f),_{\rho}^{2} - (\ln f),_{z}^{2}] + 2\rho^{-1}f(\psi,_{\rho}^{2} - \psi,_{z}^{2}), \tag{2.1.14}$$

$$(\ln Q),_{z} = \rho(\ln f),_{\rho}(\ln f),_{z} + 4\rho^{-1}f\psi,_{\rho}\psi,_{z}, \tag{2.1.15}$$

$$\psi,_{\rho\rho} - \rho^{-1}\psi,_{\rho} + \psi,_{zz} = -[\psi,_{\rho}(\ln f),_{\rho} + \psi,_{z}(\ln f),_{z}]. \tag{2.1.16}$$

We further introduce the 2×2 matrices h, S and T defined by

$$h = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} = \begin{pmatrix} f^{1/2} & f^{1/2}\psi \\ f^{1/2}\psi & f^{-1/2}\rho^{2} + f^{1/2}\psi^{2} \end{pmatrix}, \tag{2.1.17}$$

$$S = \rho h,_{\rho} h^{-1}, \quad T = \rho h,_{z} h^{-1}. \tag{2.1.18}$$

Using these matrices, we find that Eqs.(2.1.13)-(2.1.16) reduce to

$$(\rho h,_{\rho} h^{-1}),_{\rho} + (\rho h,_{z} h^{-1}),_{z} = 0, \tag{2.1.19}$$

$$(\ln Q),_{\rho} = 4(\ln h_{00}),_{\rho} - \frac{4}{\rho} + \frac{1}{\rho}\mathrm{Tr}(S^{2} - T^{2}), \tag{2.1.20}$$

$$(\ln Q),_{z} = 4(\ln h_{00}),_{z} + \frac{2}{\rho}\mathrm{Tr}(ST). \tag{2.1.21}$$

We now apply the inverse scattering method to solve Eqs.(2.1.19)-(2.1.21) with the condition

$$\det h = \rho^2. \tag{2.1.22}$$

Assuming the seed matrix $h_0 = \mathrm{diag}(1, \rho^2)$ that corresponds to the flat metric and no electromagnetic field, we find that the one-soliton solution satisfying the condition (2.1.22) is unphysical because $f < 0$. The two-soliton solution is given by

$$h_{00} = \frac{\mu_1\mu_2[\rho^2(\mu_2-\mu_1)^2(p_1p_2-q_1q_2)^2+(\rho^2+\mu_1\mu_2)^2(p_1q_2-p_2q_1)^2]}{(\mu_2-\mu_1)^2(p_1p_2\rho^2+q_1q_2\mu_1\mu_2)^2-(\rho^2+\mu_1\mu_2)^2(p_1q_2\mu_2-p_2q_1\mu_1)^2}, \tag{2.1.23}$$

$$\begin{aligned} h_{01} &= -(\mu_2-\mu_1)(\rho^2+\mu_1\mu_2) \\ &\times \frac{p_2q_2(\mu_2^2+\rho^2)(p_1^2\rho^2+q_1^2\mu_1^2)-p_1q_1(\mu_1^2+\rho^2)(p_2^2\rho^2+q_2^2\mu_2^2)}{(\mu_2-\mu_1)^2(p_1p_2\rho^2+q_1q_2\mu_1\mu_2)^2-(\rho^2+\mu_1\mu_2)^2(p_1q_2\mu_2-p_2q_1\mu_1)^2}, \end{aligned} \tag{2.1.24}$$

where

$$\begin{cases} \mu_1 = w_1 - z + \sqrt{(w_1-z)^2+\rho^2} \\ \mu_2 = w_2 - z - \sqrt{(w_2-z)^2+\rho^2}, \end{cases} \tag{2.1.25}$$

and $p_1, p_2, q_1, q_2, w_1, w_2$ are constants. Eqs.(2.1.23) and (2.1.24) lead to the results

$$f = \left[\frac{\mu_1\mu_2[\rho^2(\mu_2-\mu_1)^2(p_1p_2-q_1q_2)^2+(\rho^2+\mu_1\mu_2)^2(p_1q_2-p_2q_1)^2]}{(\mu_2-\mu_1)^2(p_1p_2\rho^2+q_1q_2\mu_1\mu_2)^2-(\rho^2+\mu_1\mu_2)^2(p_1q_2\mu_2-p_2q_1\mu_1)^2}\right]^2, \tag{2.1.26}$$

$$\begin{aligned} \psi &= -(\mu_2-\mu_1)(\rho^2+\mu_1\mu_2) \\ &\times \frac{p_2q_2(\mu_2^2+\rho^2)(p_1^2\rho^2+q_1^2\mu_1^2)-p_1q_1(\mu_1^2+\rho^2)(p_2^2\rho^2+q_2^2\mu_2^2)}{\mu_1\mu_2[\rho^2(\mu_2-\mu_1)^2(p_1p_2-q_1q_2)^2+(\rho^2+\mu_1\mu_2)^2(p_1q_2-p_2q_1)^2]}, \end{aligned} \tag{2.1.27}$$

and the integrations of Eqs.(2.1.12), (2.1.20) and (2.1.21) give

$$\begin{aligned}\chi &= (\mu_2-\mu_1)(\rho^2+\mu_1\mu_2)\\ &\times \frac{p_2q_2(p_1^2-q_1^2)\mu_1(\mu_2^2+\rho^2)-p_1q_1(p_2^2-q_2^2)\mu_2(\mu_1^2+\rho^2)}{(\mu_2-\mu_1)^2(p_1p_2\rho^2+q_1q_2\mu_1\mu_2)^2-(\rho^2+\mu_1\mu_2)^2(p_1q_2\mu_2-p_2q_1\mu_1)^2},\end{aligned} \tag{2.1.28}$$

$$Q = C_2\left[\frac{(p_1p_2-q_1q_2)^2\rho^2(\mu_2-\mu_1)^2+(p_1q_2-p_2q_1)^2(\rho^2+\mu_1\mu_2)^2}{(\mu_1^2+\rho^2)(\mu_2^2+\rho^2)}\right]^4, \tag{2.1.29}$$

where C_2 is a constant.

If we introduce the prolate spheroidal coordinates (x, y) defined by

$$\begin{cases}\rho=\sigma\sqrt{(x^2-1)(1-y^2)}\\ z-z_0=\sigma xy,\end{cases} \tag{2.1.30}$$

with

$$\sigma=\frac{w_2-w_1}{2},\quad z_0=\frac{w_2+w_1}{2}, \tag{2.1.31}$$

we then find that Eqs.(2.1.26), (2.1.28) and (2.1.29) are written as

$$f = \left[\frac{c_2^2(x^2-1)+c_4^2(1-y^2)}{(c_2x+c_1)^2-(c_3+c_4y)^2}\right]^2, \tag{2.1.32}$$

$$\chi = \frac{2(c_2c_3x-c_1c_4y)}{(c_2x+c_1)^2-(c_3+c_4y)^2}, \tag{2.1.33}$$

$$Q = C_2\left[\frac{c_2^2(x^2-1)+c_4^2(1-y^2)}{x^2-y^2}\right]^4, \tag{2.1.34}$$

where the constants c_1, c_2, c_3 and c_4 are given by

$$\begin{cases}c_1=p_1p_2+q_1q_2\\ c_2=p_1p_2-q_1q_2\\ c_3=-(p_1q_2+p_2q_1)\\ c_4=-(p_1q_2-p_2q_1).\end{cases} \tag{2.1.35}$$

In Eq.(2.1.35), we note that the following relation holds:

$$c_1^2+c_4^2=c_2^2+c_3^2. \tag{2.1.36}$$

Furthermore, if we define the relations by

$$A = \frac{c_4}{c_2}, \quad B = \frac{c_3 - c_1}{c_2}, \quad C = \frac{c_3 + c_1}{c_2}, \tag{2.1.37}$$

we then find that Eqs.(2.1.32)-(2.1.34) are rewritten as

$$\begin{aligned} f &= \left(\frac{UV - BV - CU}{UV}\right)^2, && (2.1.38)\\ \chi &= \frac{CU - BV}{UV}, && (2.1.39)\\ Q &= \left(\frac{UV - BV - CU}{x^2 - y^2}\right)^4. && (2.1.40) \end{aligned}$$

This is Bonner's solution given by Eqs.(2.1.1)-(2.1.3) with $C_2 = c_2^{-8}$.

2.2 Structure of solution

Hereafter we set $z_0 = 0$ and study the structure of Bonner's solution. We first see the asymptotic behaviors of solution at the spatial infinity $\sqrt{\rho^2 + z^2} \to \infty$:

$$\begin{aligned} f &\sim 1 - \frac{2m}{\sqrt{\rho^2 + z^2}} + \frac{2m^2 + e^2}{\rho^2 + z^2} - \frac{2lez}{(\rho^2 + z^2)^{3/2}}, && (2.2.1)\\ \chi &\sim \frac{e}{\sqrt{\rho^2 + z^2}} - \frac{me}{\rho^2 + z^2} + \frac{mlz}{(\rho^2 + z^2)^{3/2}}, && (2.2.2)\\ Q &\sim 1 - \frac{(m^2 - e^2)\rho^2}{(\rho^2 + z^2)^2}, && (2.2.3) \end{aligned}$$

where m, e and l are constants given by

$$\begin{aligned} m &= 2\frac{c_1}{c_2}\sigma, && (2.2.4)\\ e &= 2\frac{c_3}{c_2}\sigma, && (2.2.5)\\ l &= -\frac{c_4}{c_2}\sigma. && (2.2.6) \end{aligned}$$

We note that Eq.(2.1.36) gives the relation

$$m^2 - e^2 = 4(\sigma^2 - l^2), \tag{2.2.7}$$

and find that the field described by the solution has a dipole moment.

We next present the behaviors of solution around the symmetry axis defined by $\rho = 0$. In the following investigation, we assume that $\sigma > 0$ and $l > 0$ without loss of generality. Dividing the axis into three regions, we have

i) $z > \sigma$ region

$$f = \frac{16(\sigma^2 - z^2)^2}{[(e-2l)^2 - (m+2z)^2]^2}, \tag{2.2.8}$$

$$Q = 1, \tag{2.2.9}$$

$$\chi = \frac{-4(lm + ez)}{(e-2l)^2 - (m+2z)^2}, \tag{2.2.10}$$

ii) $-\sigma < z < \sigma$ region

$$f = \frac{16 l^4 (\sigma^2 - z^2)^2}{(m^2\sigma^2 - e^2\sigma^2 + 4m\sigma^3 + 4\sigma^4 + 4el\sigma z - 4l^2 z^2)^2}, \tag{2.2.11}$$

$$Q = \frac{l^8}{\sigma^8}, \tag{2.2.12}$$

$$\chi = \frac{4\sigma(e\sigma^2 + lmz)}{m^2\sigma^2 - e^2\sigma^2 + 4m\sigma^3 + 4\sigma^4 + 4el\sigma z - 4l^2 z^2}, \tag{2.2.13}$$

iii) $z < -\sigma$ region

$$f = \frac{16(\sigma^2 - z^2)^2}{[(e+2l)^2 - (m-2z)^2]^2}, \tag{2.2.14}$$

$$Q = 1, \tag{2.2.15}$$

$$\chi = \frac{4(lm + ez)}{(e+2l)^2 - (m-2z)^2}. \tag{2.2.16}$$

We finally study the singularity structure of solution in the prolate spheroidal coordinates (x, y). Bonner's solution with the constants given by Eqs.(2.2.4)-(2.2.6) is written in the coordinates (x, y) as

$$\chi = \frac{4(e\sigma x + lmy)}{(2\sigma x + m)^2 - (2ly - e)^2}, \tag{2.2.17}$$

$$f = \frac{16[\sigma^2(x^2-1) + l^2(1-y^2)]^2}{[(2\sigma x + m)^2 - (2ly - e)^2]^2}, \tag{2.2.18}$$

$$Q = \frac{[\sigma^2(x^2-1) + l^2(1-y^2)]^4}{\sigma^8 (x^2 - y^2)^4}. \tag{2.2.19}$$

The curvature invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ is given by

$$\begin{aligned} R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} &= \frac{4096\sigma^{12}(x^2-y^2)^5}{[(m+2\sigma x)^2-(e-2ly)^2]^8[\sigma^2(x^2-1)+l^2(1-y^2)]^8} \\ &\times \Big(\text{polynomial of } x, y\Big). \end{aligned} \tag{2.2.20}$$

The points $(x, y) = (\pm 1, \pm 1)$ are not singularities unless $(m\pm 2\sigma)^2-(e\pm 2l)^2 = 0$ because

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{16384(5l^4-6l^2\sigma^2+3\sigma^4)}{[(m+2\sigma)^2-(e-2l)^2]^4}, \quad \text{at } (x,y)=(1,1) \tag{2.2.21}$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{16384(5l^4-6l^2\sigma^2+3\sigma^4)}{[(m+2\sigma)^2-(e+2l)^2]^4}, \quad \text{at } (x,y)=(1,-1) \tag{2.2.22}$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{16384(5l^4-6l^2\sigma^2+3\sigma^4)}{[(m-2\sigma)^2-(e-2l)^2]^4}, \quad \text{at } (x,y)=(-1,1) \tag{2.2.23}$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{16384(5l^4-6l^2\sigma^2+3\sigma^4)}{[(m-2\sigma)^2-(e+2l)^2]^4}. \quad \text{at } (x,y)=(-1,-1) \tag{2.2.24}$$

However, we find that the points $(x, y) = (-1, \pm 1)$ are singularities when $(m, e) = (2\sigma, \pm 2l)$. In this case, the curvature invariant at $y = \pm 1$ is given by

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{64(5l^4-12l^2\sigma^2+12\sigma^4+6l^2\sigma^2x-12\sigma^4x+3\sigma^4x^2)}{\sigma^8(x+1)^8}. \tag{2.2.25}$$

We also find that the points $(x, y) = (1, \pm 1)$ are singularities when $(m, e) = (-2\sigma, \pm 2l)$. In this case, the curvature invariant at $y = \pm 1$ is given by

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{64(5l^4-12l^2\sigma^2+12\sigma^4-6l^2\sigma^2x+12\sigma^4x+3\sigma^4x^2)}{\sigma^8(x-1)^8}. \tag{2.2.26}$$

In the more general case, singularities exist at the points where the following relations hold:

$$y = -\frac{\sigma}{l}x - \frac{m-e}{2l}, \tag{2.2.27}$$

$$y = \frac{\sigma}{l}x + \frac{m+e}{2l}. \tag{2.2.28}$$

In the case $\sigma > l$, if

$$m \pm e > -2(\sigma - l), \tag{2.2.29}$$

there are no singularities in the region $x > 1$ but singularities in the region $x < -1$, while if

$$m \pm e < 2(\sigma - l), \tag{2.2.30}$$

there are no singularities in the region $x < -1$ but singularities in the region $x > 1$. In the case $\sigma < l$, there are singularities in both regions.

2.3 Definitions of mass and charge

From the asymptotic behaviors (2.2.1) and (2.2.2) at the spatial infinity, we find that Bonner's solution describes the field created by a total mass m and electric charge e. In order to see where the individual masses and charges exist, let us calculate the Komar integrals over isolated three dimensional surfaces. The Komar charge is defined by the flux integral over a three dimensional surface S_i as

$$e_i = \frac{1}{4\pi} \oint_{S_i} F^{0l} \sqrt{-g}\, ds_l, \tag{2.3.1}$$

where ds_l is the surface element of S_i. In the metric (2.1.4), we can write the integral (2.3.1) explicitly as

$$e_i = -\frac{1}{4\pi} \oint_{S_i} (\rho f^{-1} \chi_{,\rho}\, ds_\rho + \rho f^{-1} \chi_{,z}\, ds_z), \tag{2.3.2}$$

and using the relation (2.1.12), we have

$$e_i = \frac{1}{4\pi} \oint_{S_i} (\psi_{,z}\, ds_\rho - \psi_{,\rho}\, ds_z). \tag{2.3.3}$$

If we consider S_i as a cylinder specified by a lower base at $z = z_d$, upper base at $z = z_u$ and side at $\rho = \rho_s$, we can calculate the integral (2.3.3) as

$$e_i = \frac{1}{2}[\psi(0, z_u) - \psi(0, z_d)]. \tag{2.3.4}$$

The Komar mass is defined by the surface integral of the covariant derivative of the time-like Killing vector ξ^μ over S_i. Taking account of the contribution of electrostatic field to the surface integral, we have [19]

$$m_i = \frac{1}{4\pi} \oint_{S_i} (\xi^{0;l} + \chi F^{0l}) \sqrt{-g}\, ds_l. \tag{2.3.5}$$

In the metric (2.1.4), the integral (2.3.5) is written as

$$m_i = \frac{1}{8\pi} \oint_{S_i} [\rho f^{-1}(f-\chi^2)_{,\rho}\, ds_\rho + \rho f^{-1}(f-\chi^2)_{,z}\, ds_z]. \tag{2.3.6}$$

Substituting the solutions (2.1.26) and (2.1.28) into the integral (2.3.6) and performing the integration, we obtain the expression

$$m_i = \frac{1}{8\pi} \oint_{S_a} (K_{,z}\, ds_\rho - K_{,\rho}\, ds_z), \tag{2.3.7}$$

where K is the function of ρ and z given by

$$\begin{aligned} K &= (\mu_2-\mu_1)(\rho^2+\mu_1\mu_2) \\ &\times \frac{\rho^2(\mu_2^2-\mu_1^2)(p_1^2p_2^2-q_1^2q_2^2)+(\rho^4-\mu_1^2\mu_2^2)(p_1^2q_2^2-p_2^2q_1^2)}{\mu_1\mu_2[\rho^2(\mu_2-\mu_1)^2(p_1p_2-q_1q_2)^2+(\rho^2+\mu_1\mu_2)^2(p_1q_2-p_2q_1)^2]}. \end{aligned} \tag{2.3.8}$$

If we consider the surface S_i as the same cylinder as mentioned above, we have

$$m_i = \frac{1}{4}[K(0,z_u) - K(0,z_d)]. \tag{2.3.9}$$

When we set $(z_u, z_d) = (+\infty, -\infty)$ in the definitions (2.3.4) and (2.3.9), we have $e_{total} = e$ and $m_{total} = m$, respectively, which coincide with the definitions given by the asymptotic behaviors (2.2.1) and (2.2.2) of solution. Setting $(z_u, z_d) = (+\infty, +\sigma+0), (+\sigma-0, -\sigma+0)$ and $(-\sigma-0, -\infty)$, we have $e_i = 0$ and $m_i = 0$, which show that no charge and mass exist in these regions. Finally, setting $(z_u, z_d) = (+\sigma+0, +\sigma-0)$ and $(z_u, z_d) = (-\sigma+0, -\sigma-0)$, we conclude that the charge and mass exist at the points $z = +\sigma$ and $z = -\sigma$ on the axis defined by $\rho = 0$. The definition (2.3.4) gives the charges at $z = +\sigma$ and $z = -\sigma$ as

$$e_{+\sigma} = \frac{e}{2} + \frac{m^2 - e^2 + 2m\sigma}{4l}, \tag{2.3.10}$$

$$e_{-\sigma} = \frac{e}{2} - \frac{m^2 - e^2 + 2m\sigma}{4l}, \tag{2.3.11}$$

respectively. The definition (2.3.9) gives the masses at $z = +\sigma$ and $z = -\sigma$ as

$$m_{+\sigma} = \frac{m}{2} + \frac{e\sigma}{2l}, \tag{2.3.12}$$

$$m_{-\sigma} = \frac{m}{2} - \frac{e\sigma}{2l}, \tag{2.3.13}$$

respectively. We note here that the relations $e_{total} = e_{+\sigma} + e_{-\sigma}$ and $m_{total} = m_{+\sigma} + m_{-\sigma}$ hold.

2.4 Equilibrium condition

As is seen in the preceding subsections, Bonner's solution describes the gravitational and electric fields created by two charged masses located at the points $z = +\sigma$ and $z = -\sigma$ on the symmetry axis. In the general case, there exists a strut between these points. In the case where there is no strut, two charged masses are in a static equilibrium in which Coulomb force precisely balances gravitational force. As the strut causes a conical singularity on the axis, let us calculate the quantity P_0 defined by

$$P_0^2 = \lim_{\rho \to 0} \left(\frac{g_{\phi\phi}}{\rho^2 g_{\rho\rho}} \right) = \lim_{\rho \to 0} Q^{-1}. \tag{2.4.1}$$

When $P_0 = 1$ the axis is spatially Euclidean and $P_0 \neq 1$ expresses a deviation from the spatially Euclidean metric.

From the behaviors of solution around the axis given in Eqs.(2.2.8)-(2.2.16), we find that the axis is spatially Euclidean and there is no conical singularity in the regions $z > +\sigma$ and $z < -\sigma$. In the region $-\sigma < z < +\sigma$, the axis is spatially Euclidean and there is no strut if the relation $l = \sigma$ holds. In this case, the equilibrium realizes in Bonner's solution. If we set $\sigma = l$ in the relation (2.2.7), we have $m = \pm e$. Therefore, in the equilibrium case, the charges and masses of each body become

$$e_{+\sigma} = e,\ e_{-\sigma} = 0,\ m_{+\sigma} = m,\ m_{-\sigma} = 0, \tag{2.4.2}$$

or

$$e_{+\sigma} = 0,\ e_{-\sigma} = e,\ m_{+\sigma} = 0,\ m_{-\sigma} = m. \tag{2.4.3}$$

We find that one electric charge and mass disappear in the equilibrium case. In this case, the solution reduces to

$$\chi = \frac{(ex + my)}{\sigma(x^2 - y^2) + mx + ey}, \tag{2.4.4}$$

$$f = \frac{\sigma^2 (x^2 - y^2)^2}{[\sigma(x^2 - y^2) + mx + ey]^2}, \tag{2.4.5}$$

$$Q = 1. \tag{2.4.6}$$

Taking account of the relation $m = \pm e$, we have

$$\chi = \frac{\pm m}{\sigma(x \mp y) + m}, \tag{2.4.7}$$

$$f = \frac{\sigma^2 (x \mp y)^2}{[\sigma(x \mp y) + m]^2}. \tag{2.4.8}$$

We find that $f = (1 \mp \chi)^2$ and the solution becomes the one-body type of Majumdar-Papapetrou solution. We can also find that the solution in this case is reduced to the extremal Reissner-Nordström solution.

3 The double Reissner-Nordström solution

3.1 Derivation of solution

The Reissner-Nordström solution is obtain by assuming a relationship $f(\chi)$ in Einstein-Maxwell equations (2.1.8)-(2.1.11). This assumption leads to the relation

$$f = 1 - 2c\chi + \chi^2, \tag{3.1.1}$$

and reduces Eqs.(2.1.8)-(2.1.11) to

$$\chi_{,\rho\rho} + \rho^{-1}\chi_{,\rho} + \chi_{,zz} = 2f^{-1}(\chi - c)(\chi_{,\rho}^2 + \chi_{,z}^2), \tag{3.1.2}$$

$$(\ln Q)_{,\rho} = 2\rho f^{-2}(c^2 - 1)(\chi_{,\rho}^2 - \chi_{,z}^2), \tag{3.1.3}$$

$$(\ln Q)_{,z} = 4\rho f^{-2}(c^2 - 1)\chi_{,\rho}\chi_{,z}, \tag{3.1.4}$$

where c is an arbitrary constant. Introducing a function $R(\rho, z)$ with constants b and $a = bc$ by

$$\chi = \frac{b}{R + a}, \tag{3.1.5}$$

we write the function f as

$$f = \frac{R^2 - d^2}{(R + a)^2}, \tag{3.1.6}$$

and Eq.(3.1.2) as

$$R_{,\rho\rho} + \rho^{-1}R_{,\rho} + R_{,zz} = 2R(R^2 - d^2)^{-1}(R_{,\rho}^2 + R_{,z}^2), \tag{3.1.7}$$

where $d^2 = a^2 - b^2$. Further introducing a function $\bar{f}$ by

$$R = d\frac{1+\bar{f}}{1-\bar{f}}, \tag{3.1.8}$$

we find that this relation transforms Eq.(3.1.7) into a linear differential equation

$$(\ln \bar{f}),_{\rho\rho} + \rho^{-1}(\ln \bar{f}),_{\rho} + (\ln \bar{f}),_{zz} = 0, \tag{3.1.9}$$

and Eq.(3.1.3) and (3.1.4) into

$$\begin{aligned} (\ln Q),_{\rho} &= \frac{\rho}{2}[(\ln \bar{f}),_{\rho}^{2} - (\ln \bar{f}),_{z}^{2}], && (3.1.10)\\ (\ln Q),_{z} &= \rho(\ln \bar{f}),_{\rho}(\ln \bar{f}),_{z}\,. && (3.1.11) \end{aligned}$$

The expressions (3.1.9)-(3.1.11) are the same as those for the vacuum Einstein equation in the metric (2.1.4) with $\bar{f}$ instead of f. Applying the inverse scattering method to solve Eqs. (3.1.9)-(3.1.11), we have the 2-soliton solution given by

$$\begin{aligned} \bar{f} &= -\frac{\mu_1\mu_2}{\rho^2}, && (3.1.12)\\ Q &= \frac{\rho^2(\mu_1-\mu_2)^2}{(\mu_1^2+\rho^2)(\mu_2^2+\rho^2)}, && (3.1.13) \end{aligned}$$

with

$$\begin{cases} \mu_1 = z_0 - z - d + \sqrt{(z_0 - z - d)^2 + \rho^2}, \\ \mu_2 = z_0 - z + d - \sqrt{(z_0 - z + d)^2 + \rho^2}, \end{cases} \tag{3.1.14}$$

where z_0 is a constant. In the prolate spheroidal coordinates (x, y) defined by

$$\begin{cases} \rho = d\sqrt{(x^2-1)(1-y^2)}, \\ z - z_0 = dxy, \end{cases} \tag{3.1.15}$$

we have

$$\begin{cases} \mu_1 = d(x-1)(1-y), \\ \mu_2 = -d(x-1)(1+y), \end{cases} \tag{3.1.16}$$

and

$$\begin{aligned}\chi &= \frac{b}{dx+a}, & (3.1.17)\\ f &= \frac{d^2(x^2-1)}{(dx+a)^2}, & (3.1.18)\\ Q &= \frac{x^2-1}{x^2-y^2}. & (3.1.19)\end{aligned}$$

These show that the 2-soliton solution for $\bar{f}$ gives the Reissner-Nordström solution.

The 4-soliton solution is given by

$$\bar{f} = \frac{\mu_1\mu_2\mu_3\mu_4}{\rho^4}, \tag{3.1.20}$$

$$Q = \frac{\rho^8[(\mu_1-\mu_2)(\mu_1-\mu_3)(\mu_1-\mu_4)(\mu_2-\mu_3)(\mu_2-\mu_4)(\mu_3-\mu_4)]^2}{(\mu_1^2+\rho^2)(\mu_2^2+\rho^2)(\mu_3^2+\rho^2)(\mu_4^2+\rho^2)(\mu_1\mu_2\mu_3\mu_4)^2C_4}, \tag{3.1.21}$$

with

$$\begin{cases}\mu_1 = z_1 - z - d_1 + \sqrt{(z_1-z-d_1)^2+\rho^2},\\ \mu_2 = z_1 - z + d_1 - \sqrt{(z_1-z+d_1)^2+\rho^2},\\ \mu_3 = z_2 - z - d_2 + \sqrt{(z_2-z-d_2)^2+\rho^2},\\ \mu_4 = z_2 - z + d_2 - \sqrt{(z_2-z+d_2)^2+\rho^2},\end{cases} \tag{3.1.22}$$

and

$$C_4 = 16[(z_1-z_2)^2-(d_1-d_2)^2]^2, \tag{3.1.23}$$

where z_1, z_2, d_1, d_2 are constants. For the quantities f and χ, we have

$$\begin{aligned}f &= \frac{4d^2\rho^4\mu_1\mu_2\mu_3\mu_4}{[(d+a)\rho^4+(d-a)\mu_1\mu_2\mu_3\mu_4)]^2}, & (3.1.24)\\ \chi &= \frac{b(\rho^4-\mu_1\mu_2\mu_3\mu_4)}{(d+a)\rho^4+(d-a)\mu_1\mu_2\mu_3\mu_4}. & (3.1.25)\end{aligned}$$

The 4-soliton solution obtained here is the $N=2$ case of the axisymmetric N-Reissner-Normström solution and the so-called double Reissner-Nordström solution.

3.2 Structure of solution

In order to see the structure of the double Reissner-Nordström solution obtained in the preceding subsection, we first study the asymptotic behaviors of solution at the spatial infinity. In the spatial infinity $\sqrt{\rho^2+z^2} \to \infty$, we have

$$f \sim 1 - \frac{2a(d_1+d_2)}{d\sqrt{\rho^2+z^2}} + \frac{(2a^2+b^2)(d_1+d_2)^2}{d^2(\rho^2+z^2)} - \frac{2a(d_1z_1+d_2z_2)z}{d(\rho^2+z^2)^{3/2}}, \tag{3.2.1}$$

$$\chi \sim \frac{b(d_1+d_2)}{d\sqrt{\rho^2+z^2}} - \frac{ab(d_1+d_2)^2}{d^2(\rho^2+z^2)} + \frac{b(d_1z_1+d_2z_2)z}{d(\rho^2+z^2)^{3/2}}, \tag{3.2.2}$$

$$Q \sim 1 - \frac{(d_1+d_2)^2\rho^2}{(\rho^2+z^2)^2}. \tag{3.2.3}$$

Hereafter we assume that $z_2+d_2 > z_2-d_2 > z_1+d_1 > z_1-d_1$ without loss of generality. The behaviors of solution around the axis defined by $\rho = 0$ are obtained in the following regions as

i) $z > z_2+d_2$ region

$$f \sim \rho^0, \quad Q = 1, \quad \chi \sim \rho^0, \tag{3.2.4}$$

ii) $z_2+d_2 > z > z_2-d_2$ region

$$f \sim \rho^2, \quad Q \sim \rho^2, \quad \chi = \frac{b}{d+a}, \tag{3.2.5}$$

iii) $z_2-d_2 > z > z_1+d_1$ region

$$f \sim \rho^0, \quad Q = \left[\frac{(z_1-z_2)^2-(d_1+d_2)^2}{(z_1-z_2)^2-(d_1-d_2)^2}\right]^2, \quad \chi \sim \rho^0, \tag{3.2.6}$$

iv) $z_1+d_1 > z > z_1-d_1$ region

$$f \sim \rho^2, \quad Q \sim \rho^2, \quad \chi = \frac{b}{d+a}, \tag{3.2.7}$$

v) $z_1-d_1 > z$ region

$$f \sim \rho^0, \quad Q = 1, \quad \chi \sim \rho^0. \tag{3.2.8}$$

The behaviors in the regions ii) and iv) are the same as those on the horizon in the Reissner-Nordström solution. Therefore, we conclude that there are two Reissner-Nordström black holes in these regions. The behaviors in the regions i) and v) show that the axis in these regions is spatially Euclidean. The behavior in the region iii) shows that there is a strut in this region.

3.3 Definitions of charge and mass

Let us consider the charges and masses of each Reissner-Nordström black hole. The charges are defined by Eqs.(2.3.1) and (2.3.2), and in this case the integral in Eq.(2.3.2) becomes

$$e_i = \frac{b}{8\pi d} \oint_{S_i} [\rho(\ln \bar{f})_{,\rho}\, ds_\rho + \rho(\ln \bar{f})_{,z}\, ds_z]. \tag{3.3.1}$$

The integration of Eq.(3.3.1) gives the expression

$$e_i = \frac{b}{8\pi d} \oint_{S_i} (H_{,z}\, ds_\rho + H_{,\rho}\, ds_z), \tag{3.3.2}$$

where H is the function of ρ and z given by

$$H = \mu_1 + \mu_2 + \mu_3 + \mu_4 + 4z. \tag{3.3.3}$$

If we consider the surface S_i as the same cylinder as in the subsection 2.3, we have

$$e_i = \frac{b}{4d}[H(0, z_u) - H(0, z_d)]. \tag{3.3.4}$$

As for the masses, the definitions (2.3.5) and (2.3.6) are written as

$$m_i = \frac{a}{8\pi d} \oint_{S_i} [\rho(\ln \bar{f})_{,\rho}\, ds_\rho + \rho(\ln \bar{f})_{,z}\, ds_z], \tag{3.3.5}$$

which gives the expression

$$m_i = \frac{a}{4d}[H(0, z_u) - H(0, z_d)]. \tag{3.3.6}$$

When we set $(z_u, z_d) = (+\infty, -\infty)$ in the definitions (3.3.4) and (3.3.6), we have

$$e_{total} = \frac{b(d_1 + d_2)}{d}, \quad m_{total} = \frac{a(d_1 + d_2)}{d}, \tag{3.3.7}$$

respectively, which coincide with the definitions given by the asymptotic behaviors (3.2.1) and (3.2.2). Setting $(z_u, z_d) = (+\infty, z_2{+}d_2), (z_2{-}d_2, z_1{+}d_1)$ and $(z_1 - d_1, -\infty)$, we have $e_i = 0$ and $m_i = 0$, which show that no charge and mass exist in these regions. Finally, setting $(z_u, z_d) = (z_2 + d_2, z_2 - d_2)$

and $(z_u, z_d) = (z_1 + d_1, z_1 - d_1)$, we have the charges and masses of each black hole. For the lower and upper black holes, the definition (3.3.4) gives their charges e_1 and e_2 as

$$e_1 = \frac{bd_1}{d}, \quad e_2 = \frac{bd_2}{d}, \tag{3.3.8}$$

respectively, and the definition (3.3.6) gives their masses m_1 and m_2 as

$$m_1 = \frac{ad_1}{d}, \quad m_2 = \frac{ad_2}{d}, \tag{3.3.9}$$

respectively. We note here that the relations

$$e_{total} = e_1 + e_2, \quad m_{total} = m_1 + m_2. \tag{3.3.10}$$

hold also in this case.

3.4 Equilibrium condition

The behavior of solution in the region iii) given in the subsection 3.2 shows that there exists a strut between two Reissner-Nordström black holes in general. The quantity P_0 defined by Eq.(2.4.1) becomes

$$P_0^2 = \left[\frac{(z_2 - z_1)^2 - (d_2 - d_1)^2}{(z_2 - z_1)^2 - (d_2 + d_1)^2} \right]^2. \tag{3.4.1}$$

Let us consider the equilibrium condition in this case. Imposing the condition $P_0^2 = 1$ and taking account of the assumption $z_2 - d_2 > z_1 + d_1$, we have

$$d_1 d_2 = 0. \tag{3.4.2}$$

Because $m_1 m_2 \neq 0$, Eq.(3.4.2) leads to

$$d = 0 \quad \text{or} \quad a = \pm b. \tag{3.4.3}$$

From this and the relation $m_1/e_1 = m_2/e_2 = a/b$, we find that the conditions on each mass and charge are given by

$$m_1 = \pm e_1, \quad m_2 = \pm e_2. \tag{3.4.4}$$

Conversely, when the conditions (3.4.4) hold, we have $d_1 = d_2 = d = 0$ and $P_0^2 = 1$. We find that under the conditions (3.4.4) there is no strut and the solution describes two Reissner-Nordström black holes in a static equilibrium.

In the equilibrium, by taking the limit $d_1, d_2 \to 0$ in the solutions (3.1.24) and (3.1.25), we have

$$f = \left(1 + \frac{m_1}{r_1} + \frac{m_2}{r_2}\right)^{-2}, \tag{3.4.5}$$

$$\chi = \frac{b}{a}\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)\left(1 + \frac{m_1}{r_1} + \frac{m_2}{r_2}\right)^{-1}, \tag{3.4.6}$$

where $r_1 = \sqrt{\rho^2 + (z - z_1)^2}$ and $r_2 = \sqrt{\rho^2 + (z - z_2)^2}$. This is known as the two-body type of Majumdar-Papapetrou solution. In the prolate spheroidal coordinates (x, y) defined by

$$\begin{cases} \rho = \dfrac{z_2 - z_1}{2}\sqrt{(x^2 - 1)(1 - y^2)}, \\[2ex] z = \dfrac{z_2 + z_1}{2} + \dfrac{z_2 - z_1}{2} xy, \end{cases} \tag{3.4.7}$$

we have

$$r_1 = \frac{(z_2 - z_1)(x - y)}{2}, \tag{3.4.8}$$

$$r_2 = \frac{(z_2 - z_1)(x + y)}{2}, \tag{3.4.9}$$

and

$$\chi = \frac{2b[(m_1 + m_2)x + (m_1 - m_2)y]}{a[(z_2 - z_1)(x^2 - y^2) + 2(m_1 + m_2)x + 2(m_1 - m_2)y]}, \tag{3.4.10}$$

$$f = \frac{(z_2 - z_1)^2(x^2 - y^2)^2}{[(z_2 - z_1)(x^2 - y^2) + 2(m_1 + m_2)x + 2(m_1 - m_2)y]^2}, \tag{3.4.11}$$

$$Q = 1. \tag{3.4.12}$$

This solution has event horizons at $(x, y) = (1, \pm 1)$. The curvature invariant of this solution is given by

$$\begin{aligned} R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} &= \frac{2^{11}}{[(z_2 - z_1)(x^2 - y^2) + 2(m_1 + m_2)x + 2(m_1 - m_2)y]^8} \\ &\quad \times \Big(\text{polynomial of } x, y\Big). \end{aligned} \tag{3.4.13}$$

There are singularities at the points where the coordinates (x, y) satisfy the relation

$$(z_2 - z_1)(x^2 - y^2) + 2(m_1 + m_2)x + 2(m_1 - m_2)y = 0. \tag{3.4.14}$$

We find that these points do not exist in the region $x > 1$ and exist inside the event horizons specified by $r_1 = 0$ and $r_2 = 0$. This is a very interesting two-body type of black hole solution, which was studied in detail in the paper by Hartle and Hawking [20].

4 Discussion

In this paper, we use the Komar integrals in order to define the charges and masses of each body. However, Bonner argues that it 'seems not to be allowable' to use the integrals because the surfaces S_i introduced in subsections 2.3 and 3.3 contain the points at which they meets the conical singularities [16]. In fact, the definitions of the charges and masses given in subsection 2.3 differ from those by Bonner. Bonner defines them from the asymptotic behaviors at the spatial infinity $r_{-\sigma} = \sqrt{\rho^2 + (z+\sigma)^2} \to \infty$ and $r_{+\sigma} = \sqrt{\rho^2 + (z-\sigma)^2} \to \infty$ as

$$\begin{aligned} \chi &\sim \frac{e^{(B)}_{+\sigma}}{r_{+\sigma}} + \frac{e^{(B)}_{-\sigma}}{r_{-\sigma}}, \\ f &\sim 1 - \frac{2m^{(B)}_{+\sigma}}{r_{+\sigma}} - \frac{2m^{(B)}_{-\sigma}}{r_{-\sigma}}. \end{aligned}$$

In terms of the constants introduced in subsection 2.2, the results are given by

$$\begin{aligned} e^{(B)}_{+\sigma} &= \frac{e}{2} + \frac{lm}{2\sigma}, \\ e^{(B)}_{-\sigma} &= \frac{e}{2} - \frac{lm}{2\sigma}, \\ m^{(B)}_{+\sigma} &= \frac{m}{2} + \frac{le}{2\sigma}, \\ m^{(B)}_{-\sigma} &= \frac{m}{2} - \frac{le}{2\sigma}. \end{aligned}$$

On the other hand, in the double Reissner-Nordström solution, we find that the asymptotic behaviors at the spatial infinity $r_1 = \sqrt{\rho^2 + (z - z_1)^2} \to \infty$ and $r_2 = \sqrt{\rho^2 + (z - z_2)^2} \to \infty$ are given by

$$
\begin{aligned}
\chi &\sim \frac{e_1}{r_1} + \frac{e_2}{r_2}, \\
f &\sim 1 - \frac{2m_1}{r_1} - \frac{2m_2}{r_2},
\end{aligned}
$$

where e_1, e_2, m_1 and m_2 are defined in Eqs.(3.3.8) and (3.3.9). These behaviors show that the charges and masses defined by the asymptotic behaviors of solution coincide with those by the Komar integrals. As the surfaces S_i used in the case of the double Reissner-Norström solution also contain the conical singularity, it is possible that the conical singularity does not cause the differences in the definitions of charges and masses. Further investigation on this problem should be studied in future.

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